

Multiple Voting Location Problems*

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Abstract

The facility voting location problems arise from the application of criteria derived from the voting processes concerning the location of facilities. The multiple location problems are those location problems in which the alternative solutions are sets of points. This paper extends previous results and notions on single voting location problems to the location of a set of facility points. The application of linear programming techniques to solve multiple facility voting location problems is analyzed. We propose an algorithm to solve Simpson Multiple Location problems from which the solution procedures for other problems are derived.

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1 Introduction

The multiple facility voting location problem arises from the application of the criteria derived from the voting processes to a location problem when the alternative solutions are sets of points. The application of the usual criteria in voting processes to these problems let us to obtain several concepts of solution: the Condorcet solution, the Simpson solution, the Plural solution and the Security solution. These concept solutions and the corresponding problems in single facility location have been analyzed in the literature. In single voting location problems only a point is chosen to locate the facility. The Tolerant notion of solution obtained by introducing an indifference threshold in the model in order to use semi-orders as preference structures of the users in single facility location has also been considered. The objective of the paper is to analyze the corresponding multiple facility voting location problems and to extend the known results on single location to these problems.

Next section gives a description of the application of voting criteria to the multiple facility location problems and the formulation of the corresponding model. Section three introduces the family of standard concept solutions for the multiple facility location problem. Section four analyzes the use of a linear programming approach to solve the problems. Section five provides an algorithm to solve the p -Simpson Location Problem that is experimentally analyzed on randomly generated instances. The paper ends with some conclusions and a list of references.

2 Multiple Voting Location Problems

The solutions of a multiple facility location problem are sets of location points. However, not all the sets of points are possible solutions of a voting location problem, since, in that case, the best solution would be the user location set. On the other hand, the number of facility points is usually fixed due to budgetary considerations. The solutions of a voting location problem are those locations that are not rejected by the users using a voting process.

The Condorcet solution appears by considering the voting process derived from the absolute majority criterion. An alternative solution is rejected by the users if an absolute majority prefers another solution. Thus a *Condorcet solution* of a multiple facility location problem is a set of facility points such

that no other set is preferred by more than one half of all the users. The *Simpson solution* consists of those solutions that minimize the largest number of users that prefer another possible solution. This notion is usually justified because there are often instances where there is no Condorcet solution.

Let us consider that the number of facility points is fixed at p , $p \geq 1$. If $p = 1$, the problem is a single facility location problem. If $p > 1$, it is a multiple facility location problem. The *p-Condorcet sets* are the sets of p location points such that there is no other set of p location points that is nearer to one half of the users. Here every user is assumed to always prefer a closer location for the facility and that the rejection majority is one half of the voters. Due to the nonexistence of a *p-Condorcet set* in some cases the rejection majority can be fixed at a proportion of voters different than one half of them. *p-Simpson sets* are obtained by considering the maximum rejection majority that allows a solution. However, the constraints of the solution can also be relaxed by considering an indifference threshold in the preferences of the users. It is assumed that the users prefer a solution to another one if the difference of the distances to them is larger than the threshold. *p-Tolerant Condorcet sets* are obtained by considering the minimum common threshold that allows a solution. Both relaxations can be considered at the same time, a majority rejection γ and an indifference threshold α , to deal with the $\alpha\gamma$ -Condorcet solutions. *Plural sets*, *Security sets* and *Tolerant Plural sets* are similarly defined; bearing in mind that the alternatives are compared using the simple majority rule.

2.1 Antecedents

For a review of voting location see Hansen, Thisse, Wendell, (1990). Condorcet (Hansen and Thisse, 1981), Simpson (Hakimi, 1983), Tolerant (Campos and Moreno, 2000), Plural (Wendell and McKelvey, 1981) and Security (Slater, 1975) locations have been considered in the literature regarding voting location. However, there are no previous works about the p -facility problem in voting location. Only some works in competitive location are indirectly related with these problems.

The p -Condorcet set problem is similar to a special case of competitive location problems on networks considered in Hakimi (1990). The vertex weights and edge lengths can be interpreted economically. The weight $w(v)$ of the vertex v is considered as the demand of this vertex, and the length $l(e)$

is the transportation cost along the edge e by unit of service. In the Hakimi model, at least two suppliers compete for the vertex demands. In the general model there are p established service locations $X_p = \{x_1, x_2, \dots, x_p\}$ and a competitor wants to establish r new service locations $Y_r = \{y_1, y_2, \dots, y_r\}$. There are two kinds of services: *essential services* (hospitals, fire stations, schools, ...) and *non essential services* (restaurants, shops, ...). For the essential services, the service demand does not depend on the distance from the user to the service. However, for non essential services, the demand could depend on the user distance function. *Binary preferences* are used when the user chooses the nearest service and the *proportional preferences* are used when the election depends on the relative distances. Hakimi (Hakimi, 1990) provides several models, of which the models with non essential demand and binary preferences are the most similar to the p -facility problems in voting location on networks. The demands that are at equal distance to both competitors are divided between them; each one receiving one half of the demand.

2.2 The Model

We extend the voting location model for locating one point to a set of facilities. The criterion for choosing a set of locations is the number of users that could prefer another set of locations as an alternative solution. Each user prefers the nearest facility location. The user therefore prefers the facility set nearest to the user. A user is indifferent between two facility sets if the distance from the user to the nearest facility of a set is equal to the distance from the user to the nearest facility of the other set. A facility set is rejected by a set of users when there is a sufficiently large majority that prefers a facility of another set with the same size (this set has to be the same for all the rejecting users). Different problems arise when different rejecting majorities are considered.

Let U be the set of users and let L be the set of possible facility locations on a network N . The sets U and L are usually finite and consist of vertices; i.e., $U, L \subseteq V$. Let $d(u, x)$ be the distance from the user u to the location x .

The basic model is the single Condorcet model, when only one facility point is to be located, and is formulated as follows. A location $y \in L$ is preferred to $x \in L$ if $d(u, y) < d(u, x)$. For every two locations $x, y \in L$, let $U(y \prec x)$ be the set of users that prefer the location y to the location x .

Then $U(y \prec x) = \{u \in U : d(u, y) < d(u, x)\}$. The number of users that prefer location y to location x is denoted by $W(y \prec x)$. Therefore,

$$W(y \prec x) = |U(y \prec x)| = |\{u \in U : d(u, y) < d(u, x)\}|.$$

If this number is greater than one half of the users, for some $y \in L$, the facility location x is rejected. Then, the location point $x \in L$ is a Condorcet location if there is no $y \in L$ such that $W(y \prec x) > |U|/2$.

We consider the corresponding multiple location when the points x and y of L are substituted by sets of location points with the same size. Namely, let p be the number of facilities to be established and let L^p denote the sets of p facility location points; i.e. $L^p = \{X \subseteq L : |X| = p\}$. We then define the notion $W(Y \prec X)$ and $U(Y \prec X)$ for every two sets X and Y of p location points; i.e., $X, Y \in L^p$.

The distance from a user u to a location set $Z \subseteq L$ is $d(u, Z) = \min\{d(u, z) : z \in Z\}$. The users that prefer the point $y \in L$ to the set $X \subseteq L$ are those that prefer y to any point of X . Therefore

$$U(y \prec X) = \bigcap_{x \in X} U(y \prec x) = \{u \in U : d(u, y) < d(u, X)\}.$$

The set $U(Y \prec X)$ of users that prefer the location set $Y \subseteq L$ to the location set $X \subseteq L$ is the set of users that have a point of the set Y closer than any point of the set X . Thus

$$U(Y \prec X) = \{u \in U : d(u, Y) < d(u, X)\}.$$

This is the set of users u such that for any point $x \in X$ there is a point $y \in Y$ closer to the user u than x ; this point y depends on the user u and on the location x . The set $U(Y \prec X)$ can be obtained as follows.

$$\begin{aligned} U(Y \prec X) &= \bigcup_{y \in Y} U(y \prec X) = \bigcup_{y \in Y} \bigcap_{x \in X} U(y \prec x) = \\ &= \{u \in U : \forall x \in X, \exists y \in Y : d(u, y) < d(u, x)\}. \end{aligned}$$

Finally, the number of users that prefer set Y to set X , is:

$$W(Y \prec X) = |U(Y \prec X)| = |\{u \in U : d(u, Y) < d(u, X)\}|.$$

If the location of the users at the vertices of the network is given by a weight function $w(\cdot)$, where the weight $w(v)$ of a vertex v is the number of

users located at this vertex, then the weight of any set of vertices is the sum of the weights of its elements. The total number of users is:

$$W(V) = \sum_{v \in V} w(v),$$

Therefore, the number of users that prefer Y to X is:

$$W(Y \prec X) = \sum_{v \in V(Y \prec X)} w(v),$$

where

$$V(Y \prec X) = \{v \in V : d(v, Y) < d(v, X)\}.$$

Some extensions of the models can be considered if another weight interpretation is given to the function w .

The comparison of a proposed set of locations for the facility with all the sets of locations can not be applied when defining a concept of voting solution such as the Condorcet solution. The proposed location set is compared only with the alternative location sets that have the same number of facility locations. The number of facility points is considered as the characteristic of the set that allows a comparison. Other relevant characteristics or cost constraints could be considered for a more realistic context. The usual characteristics or constraints include the number of facility locations, size, cost or capacity of the facilities, distance between facilities, etc. These characteristics could be different depending upon the application areas.

The model can also be formulated using the location sets that are able to capture the users that are assigned to a location set (Revelle, 1986). Let

$$Z(v : X) = \{z \in L : d(v, z) < d(v, X)\}$$

be the set of locations that capture the users located at v when the locations are compared with the location set X . The users located at v will prefer a facility location of the alternative location set Y to the facility locations of set X if and only if $Y \cap Z(v : X) \neq \emptyset$. Thus,

$$V(Y \prec X) = \{v \in V : Y \cap Z(v : X) \neq \emptyset\}.$$

The sets $Z(v : X)$ are the locations that have to be considered when searching for good alternatives in the comparison with facility location set X .

3 The Set Solution Notions

Several solution notions arise by considering different voting processes derived from the absolute and simple majority criteria.

3.1 The Condorcet Set

A point is a *Condorcet point* (Hansen and Thisse, 1981) if the number of users that prefer another point is less than or equal to one half of the users. Formally, $x \in L$ is a Condorcet point for the space L and the set of users U if: $W(y \prec x) \leq |U|/2, \forall y \in L$. Let C denote the set of Condorcet points stated as follows:

$$C = \{x \in L : \max_{y \in L} W(y \prec x) \leq |U|/2\}.$$

Similarly, a set X is a *p-Condorcet set* if it has p points and the number of users that prefer another set Y with p points is less than or equal to one half of the users. In order to obtain a notation totally parallel to the Condorcet point, replace the set L by L^p , the family of sets with p location points; i.e., $L^p = \{X \subseteq L : |X| = p\}$. Consider the following formal definition.

Definition 1 *A set $X \in L^p$ is a p-Condorcet set for the location space L and the set of users U if and only if $W(Y \prec X) \leq |U|/2, \forall Y \in L^p$.*

Let C_p be the set of *p-Condorcet sets*. Then

$$C_p = \{X \in L^p : \max_{Y \in L^p} W(Y \prec X) \leq |U|/2\}.$$

In general, without naming the size of the set, a location set X is a *Condorcet set* if the number of users that prefer another location set Y with the same size of location points, is not greater than one half of the users.

Definition 2 *A location set $X \subseteq L$ is a Condorcet set for L and U if and only if, $\forall Y \subseteq L: |Y| = |X| \Rightarrow W(Y \prec X) \leq |U|/2$.*

Then the Condorcet sets are those $X \subseteq L$ such that:

$$\max_{Y \in L^{|X|}} W(Y \prec X) \leq |U|/2.$$

This family of Condorcet sets is given by:

$$C_\infty = \{X \subseteq L : W(Y \prec X) \leq |U|/2, \forall Y \in L^{|X|}\} = \bigcup_{p=1}^{\infty} C_p.$$

3.2 The Simpson set

For all values $p < |L|$, the p -Condorcet set does not always exist. The rejection majority can be fixed to a number of users different to one half of them; there is always a big enough majority such that a non rejected solution exists. The Simpson solutions arise when we look for the non rejected solutions for the minimum rejection majority such that they exist. However, the Simpson solution is an interesting notion even if a Condorcet set exists.

A *Simpson point* (Hakimi, 1983) is a point such that the maximum number of users that prefer another point is the smallest possible. The *Simpson score* of a point x is this maximum number of users that prefer another point; i.e., $W^*(x) = \max_{z \in L} W(z \prec x)$; $\forall x \in L$. Then, formally, a location point $x \in L$ is a Simpson point for the location space L and for the set of users U if it minimizes the Simpson score; i.e., $x \in L$ is a Simpson point if:

$$\forall y \in L : W^*(x) = \max_{z \in L} W(z \prec x) \leq W^*(y) = \max_{z \in L} W(z \prec y).$$

Thus the Simpson problem is the minimax problem: $\min_{x \in L} \max_{z \in L} W(z \prec x)$.

The set S of Simpson points is given by:

$$S = \arg \min_{x \in L} W^*(x) = \arg \min_{x \in L} \max_{z \in L} W(z \prec x)$$

When the number of facility location points to be located is greater than one, we obtain the notion of Simpson set. Let X be a set of facility location points. Then the opposition of the set X is the best alternative set of location points with the same number of location points, when compared with it.

Definition 3 A location set $Y \subseteq L$ is an **opposition** of the set $X \subseteq L$ for the user set U if and only if $|Y| = |X|$ and

$$W(Y \prec X) \geq W(Z \prec X), \forall |Z| = |X|, Z \subseteq L.$$

The opposition of a set X coincides with the definition of X -medianoid used in Hakimi (1990). Let $Y(X)$ denote the opposition of X . Then,

$$Y(X) = \arg \max_{Y \in L^{|X|}} W(Y \prec X).$$

If we want to locate the set X , with p facility points, such that the number of users that prefer another set with p facility points is minimum, then we are using the Simpson criterion.

Definition 4 *it A location set $X \subseteq L$ is a **Simpson set** if and only if*

$$W(Y(X) \prec X) \leq W(Y(Z) \prec Z), \forall Z \in L^{|X|}.$$

The value of the maximum rejection of a set is the *Simpson score*, expressed by:

$$W^*(X) = \max_{Y \in L^{|X|}} W(Y \prec X) = W(Y(X) \prec X).$$

The concept of p -Simpson set arises when the number p of facility location points is fixed. The p -Simpson sets are the sets with p location points such that the maximum number of users that prefer another set with p location points is the smallest possible. Then $X \in L^p$ is a p -Simpson set if

$$W^*(X) = \min_{Y \in L^p} W^*(Y).$$

Definition 5 *A location set $X \in L^p$ is a **p -Simpson set** if and only if*

$$X = \arg \min_{Z \in L^p} \max_{Y \in L^p} W(Y \prec Z).$$

Let S_p denote the set of p -Simpson sets. Then

$$S_p = \left\{ X \in L^p : W(Y(X) \prec X) = \min_{Z \in L^p} W(Y(Z) \prec Z) \right\}.$$

These sets are named the *centroids* in Hakimi (1990) because they are the solutions of the following minimax problem:

$$\min_{X \in L^p} \max_{Y \in L^p} W(Y \prec X).$$

For any value of p , a p -Condorcet set does not always exist and there is at least one p -Simpson set. Not every p -Condorcet set is a p -Simpson set and not every p -Simpson set is a p -Condorcet set. However, if X is a p -Simpson set and its score is less than one half of the users then it is also a p -Condorcet set. Finally, if there exists a p -Condorcet set then every p -Simpson set is also a p -Condorcet set.

3.3 The Plural set

The concept of plural solution arises when the absolute majority in the Condorcet notion is replaced by the simple majority. Then a proposed facility location is rejected if there is another alternative location preferred by a simple majority of users. A point x is a *plural location* (Wendell and McKelvey, 1981) if there is no other location point y that is preferred for more users than the number of users that prefer the point x . The location $x \in L$ is a plural solution for the space L and for the user set U if: $W(y \prec x) \leq W(x \prec y)$, $\forall y \in L$. Let P denote the set of plural points. Then,

$$P = \{x \in L : W(y \prec x) \leq W(x \prec y), \forall y \in L\}.$$

This notion is then also extended to facility location sets. Therefore, a location point set X is a *plural solution* if the number of users that prefer another location point set Y , with the same number of location points, is not greater than the number of users that prefer the set X . Formally, consider a given number p for the size of the facility sets, the p -plural set is defined as follows.

Definition 6 *A set $X \in L^p$ is a p -plural set for the location space L and for the user set U if $W(Y \prec X) \leq W(X \prec Y)$, $\forall Y \in L^p$.*

Let P_p be the set of p -plural sets. Then,

$$P_p = \{X \in L^p : W(Y \prec X) \leq W(X \prec Y), \forall Y \in L^p\}$$

and the plural sets are the p -plural sets for some p . The plural sets are the sets of location points such that there is no other location set with the same size that is preferred by a simple majority of users. Formally,

Definition 7 *A location point set $X \subseteq L$ is a plural set if and only if*

$$\forall Y \subseteq L : |Y| = |X| \Rightarrow W(Y \prec X) \leq W(X \prec Y).$$

In parallel with the notion of Simpson set from the Condorcet set, the notion of *security set* or *Copeland set* (Slater, 1975) arises when the absolute majority in the Simpson notion is replaced by the simple majority. The security score is defined in a similar way to the Simpson score. Let the security score of a set X be

$$W'(X) = \max_{Y \in L^{|X|}} [W(Y \prec X) - W(X \prec Y)].$$

Note that if a set does not have a positive security score, then it is a plural set. The set of p -plural sets is:

$$P_p = \{X \in L^p : W'(X) \leq 0\}.$$

Thus, the set with the smallest security score for all the sets with the same number of location points is a security set or Copeland set.

Definition 8 *A location point set $X \subseteq L$ is a **Security set** if and only if,*

$$W'(X) = \min_{Y \in L^{|X|}} W'(Y)$$

Given the size p for the facility set, the p -security sets are the sets X in L^p with the smallest security score.

Definition 9 *A location point set X is a **p -Security set** if and only if,*

$$X = \arg \min_{Y \in L^p} W'(Y).$$

Let K_p denote the set of the p -security sets. Then

$$K_p = \{X \in L^p : W'(X) = \min_{Y \in L^p} W'(Y)\}.$$

An opposition of a location point set X using the simple majority is a set Y with the same number of facility location points and with a security score equal to $W'(X)$. Thus, the opposition of a location point set X using the simple majority is defined by:

$$Y'(X) = \arg \max_{Y \in L^{|X|}} [W(Y \prec X) - W(X \prec Y)].$$

3.4 The Tolerant set

The concepts of tolerant solution are obtained by using the minimum indifference common threshold for the preference of the users that allows a Condorcet or a plural solution to exist. Assume that each user is indifferent between two facility location points if the difference of the distances from the user to them is not greater than a positive threshold α . We are considering a new preference system using the same indifference threshold α for all

the users. The following notions of α -Condorcet, α -Simpson, α -Plural and α -Security sets are then obtained using this indifferent threshold.

With this new preference model, a user u prefers a location point x to another location y if $d(u, x) < d(u, y) - \alpha$. Let the set of users that prefer a facility location point of the set Y to any facility location point of the set X be $U_\alpha(Y \prec X) = \{u \in U : d(u, Y) < d(u, X) - \alpha\}$. The number of users that prefer Y to X is: $W_\alpha(Y \prec X) = |U_\alpha(Y \prec X)|$. Thus, the α -Simpson score and the α -security score are defined by: $W_\alpha^*(X) = \max_{|Y|=|X|} W_\alpha(Y \prec X)$. and $W'_\alpha(X) = \max_{|Y|=|X|} [W_\alpha(Y \prec X) - W_\alpha(X \prec Y)]$.

The definitions of α -Condorcet, α -Simpson, α -plural and α -security sets are obtained by replacing the Simpson score W^* by the α -Simpson score W_α^* in the definitions of Condorcet and Simpson sets, and replacing the security score W' by the α -security score W'_α in the definitions of plural and security sets, respectively.

For a fixed value of p , the α -Condorcet, α -Simpson, α -plural and α -security sets, respectively denoted by C_p^α , S_p^α , P_p^α and K_p^α are given by:

$$\begin{aligned} C_p^\alpha &= \{X \in L^p : W_\alpha^*(X) \leq |U|/2\}. \\ S_p^\alpha &= \{X \in L^p : X = \arg \min_{Y \in L^p} W_\alpha^*(Y)\}. \\ P_p^\alpha &= \{X \in L^p : W'_\alpha(X) \leq 0\}. \\ K_p^\alpha &= \{X \in L^p : X = \arg \min_{Y \in L^p} W'_\alpha(Y)\}. \end{aligned}$$

Since the value of α is interpreted as a tolerance distance for the feeling by the users that two locations are indifferent, we are asking all the users for a same tolerance threshold. Then the minimum value of α such that a solution exists provides the tolerance necessary to arrive at a compromise solution. The corresponding notions are qualified with the word *tolerant* (Campos and Moreno, 2000); so we have the notions of Tolerant Condorcet Set and Tolerant Plural Set. Therefore, the p -tolerant Condorcet sets and the p -tolerant plural sets are the sets corresponding to the minimum values of α such that there exists a p -Condorcet set and a p -plural set, respectively.

Let the corresponding minimum distance be called the tolerance distance. Thus, for the Condorcet solution notion, the tolerance distance is given by: $\tau = \arg \min\{\alpha \geq 0 : C_p^\alpha \neq \emptyset\}$. Then, the p -tolerant Condorcet sets, TC_p , are the p -Condorcet sets for the tolerance distance, i.e., the sets in C_p^τ . Analogously, for the plural solution notion, the tolerance distance is

$\tau = \arg \min\{\alpha \geq 0 : P_p^\alpha \neq \emptyset\}$. Thus the p -tolerant plural sets TP_p are the sets in P_p^τ , for this tolerance distance.

3.5 Other Extensions

Other extensions of these notions are obtained by considering a rejection majority different of the standard majorities (absolute or simple), and introducing a new parameter γ (Campos and Moreno, 2001). We can consider this parameter γ different to $1/2$ for the absolute majority rule to introduce the γ -Condorcet and the γ -Simpson solution notions and also different to 0 for the simple majority rule to introduce the γ -plural and the γ -security solution notions. These extensions can also be considered at the same time of the threshold α .

For the Condorcet notion, the threshold α and the parameter γ provide the notion of $\alpha\gamma$ -Condorcet set. A location point set X is an $\alpha\gamma$ -Condorcet set if

$$\max_{Y \in L^{|X|}} W_\alpha(Y \prec X) \leq \gamma|U|.$$

Let $\alpha\gamma C$ denote the $\alpha\gamma$ -Condorcet sets.

The notion of α -Simpson set also arises from this solution notion by considering the minimum value for γ such that a solution exists. For fixed values of p and α , the α -Simpson sets with p facility location points are the $\alpha\gamma$ -Condorcet sets with p facility location points for the smallest value γ such that this set is not empty, that is, $\gamma^* = \min\{\gamma : \alpha\gamma C \neq \emptyset\}$.

Parallel extensions and observations can be considered for the notions of plural and security sets using the simple majority instead of the absolute majority; i.e., $W_\alpha(Y \prec X) - W_\alpha(X \prec Y)$ instead of $W_\alpha(Y \prec X)$.

4 Integer Programming Formulation

This section analyzes the use of integer linear programming techniques for solving the above multiple voting location problems. The initial purpose is to formulate the p -Simpson set problem in terms of integer linear programming. A p -Condorcet set can be obtained, if it exists, by solving the p -Simpson set problem. The procedure can be easily extended to find $\alpha\gamma$ -Condorcet sets

and also adapted to solve the multiple voting location problems. This approach follows the one used by Dobson and Karmarkar when they formulated the competitive location problem as a linear programming problem (Dobson and Karmarkar, 1987).

4.1 The Three Stage Optimization Process

The multiple voting location problem is formulated as a simultaneous three-stage optimization process. The processes includes:

The user selection problem. Given the proposed location set and the alternative location set, choose the preferred service point.

The alternative location set problem. Given the proposed location set, select the alternative set of location preferred for the maximum number of users.

The proposed location set problem. Determine the location set such that the best alternative location is preferred by the minimum number of users.

Let $m = |L|$ be the number of possible facility locations and let $n = |U|$ be the number of user locations. Let $d_{ki} = d(u_k, f_i)$ be the distance between the k -th user location u_k and the i -th facility point f_i . In addition, let w_k be the number of users located at u_k . Let $K = \{1, 2, \dots, n\} = [1..n]$ denote the index set for the user locations and $I = \{1, 2, \dots, m\} = [1..m]$ denote the index set for the facility locations. A set Z of location points is identified by a binary vector z with size m . This vector is $z = (z_1, z_2, \dots, z_m) = (z_i : i \in [1..m])$ where $z_i = 1$ if $f_i \in Z$ and $z_i = 0$ if $f_i \notin Z$, and then $Z = \{f_i \in L : z_i = 1\}$.

The decision variables in the proposed and alternative location problems are the sets X and Y that are the represented by the corresponding m -vectors x and y of 0-1 decision variables. For the users problem, let z_{ki} be the 0-1 decision variables indicating whether the users located at the k user location u_k prefer the location f_i for the facility. However, in each user problem, the sets X and Y are data that are represented by corresponding m -vectors of 0-1 values \bar{x} and \bar{y} for the variables x and y . For the alternative location problem we have an m -vector of values \bar{x} and an m -vector of variables y .

4.2 The Users Selection Problem

Consider the Users Selection Optimization (USO) Problem. This problem consists of selecting the nearest facility to each user, given a proposed location set corresponding to a vector of values \bar{x} for the x variables, and an alternative location set corresponding to a vector of values \bar{y} for the y variables. Let $USO(\bar{x}, \bar{y})$ denote this problem.

Proposition 10 *For every m -vectors \bar{x} and \bar{y} , the Users Selection Optimization Problem $USO(\bar{x}, \bar{y})$ is a linear problem with nm binary variables and $n(m + 1)$ constraints.*

Proof. The optimization problem $USO(\bar{x}, \bar{y})$ can be formulated as follows.

$$\begin{aligned}
 & \text{Minimize} && \sum_{k=1}^n \sum_{i=1}^m d_{ki} w_k z_{ki} \\
 & \text{Subject to:} && \sum_{i=1}^m z_{ki} = 1 && k \in [1..n] \\
 & && z_{ki} \leq \bar{x}_i + \bar{y}_i && k \in [1..n]; i \in [1..m] \\
 & && z_{ki} \in \{0, 1\} && k \in [1..n]; i \in [1..m]
 \end{aligned} \tag{1}$$

The objective function of this problem represents the total distance of the users to arrive at the corresponding facility point. The constraints state that each user has to go to one location in the proposed location set or in the alternative location set. Since $|K| = n$ and $|I| = m$, the problem has nm binary variables and $n(m + 1)$ constraints.

Proposition 11 *For every m -vectors \bar{x} and \bar{y} , the Users Selection Optimization Problem $USO(\bar{x}, \bar{y})$ is separable into n linear optimization problems with m binary variables and $m + 1$ constraints.*

Proof. The $USO(\bar{x}, \bar{y})$ problem is separable in each user location. The problem of each user located at u_k is:

$$\begin{aligned}
& \text{Minimize} && \sum_{i=1}^m d_{ki} w_k z_{ki} \\
& \text{Subject to:} && \sum_{i=1}^m z_{ki} = 1 \\
& && z_{ki} \leq \bar{x}_i + \bar{y}_i \quad i \in [1..m] \\
& && z_{ki} \in \{0, 1\} \quad i \in [1..m]
\end{aligned} \tag{2}$$

Since $|I| = m$ this problem has m binary variables and $m + 1$ constraints.

Let $USO_k(\bar{x}, \bar{y})$ denote the User Selection Optimization problem of any user located at u_k . Given the proposed and alternative locations given by \bar{x} and \bar{y} , this problem consists in selecting the preferred location among them.

Proposition 12 *The users problem $USO(\bar{x}, \bar{y})$ has a direct solution from the m -vectors \bar{x} and \bar{y} .*

Proof. Each user problem $USO_k(\bar{x}, \bar{y})$ has a direct solution using the coefficients c_{ij}^k given, $\forall k \in [1..n]$ and $\forall i, j \in [1..m]$, by $c_{ij}^k = 1$, if $d_{ki} < d_{kj}$ and $c_{ij}^k = 0$, if $d_{ki} \geq d_{kj}$. A user at point u_k prefers the location f_i to the location f_j if and only if $c_{ij}^k = 1$. If $i_k = \arg \min\{d_{kj} : \bar{x}_j = 1 \text{ or } \bar{y}_j = 1\}$. Then the solution is given by: $z_{ki_k} = 1$. Therefore, if $z_{ki} = 1$ then $c_{ji}^k = 0$, for all $j \neq i$ such that $\bar{x}_j + \bar{y}_j = 1$. In other words, if for some j we have $\bar{x}_j + \bar{y}_j = 1$ and $c_{ji}^k = 1$ then $z_{ki} = 0$. The direct solution of $USO(\bar{x}, \bar{y})$ is obtained by combining all solutions of the problems $USO_k(\bar{x}, \bar{y})$, for $k \in [1..n]$.

Proposition 13 *For every m -vectors \bar{x} and \bar{y} , the users problem $USO(\bar{x}, \bar{y})$ can be solved by a linear system with nm binary variables and $n(2m + 1)$ constraints.*

Proof. The direct solution of the above proposition is obtained from the linear constraint $z_{ki} + c_{ji}^k(\bar{x}_j + \bar{y}_j) \leq 1$. Thus, the optimal solution of

$USO_k(\bar{x}, \bar{y})$ is the solution of the system in the z variables.

$$\begin{aligned}
\sum_{i=1}^m z_{ki} &= 1 \\
z_{ki} &\leq (\bar{x}_i + \bar{y}_i) & i \in [1..m] \\
z_{ki} &\leq 1 - c_{ji}^k(\bar{x}_j + \bar{y}_j) & i, j \in [1..m] \\
z_{ki} &\in \{0, 1\} & i \in [1..m]
\end{aligned} \tag{3}$$

Since $|I| = m$, this problem has m variables and $m^2 + m + 1$ linear constraints. Observe that vectors \bar{x} and \bar{y} are data for these problems. Note that the last set of constraints forces z_{ki} to be zero if $c_{ji}^k(\bar{x}_j + \bar{y}_j) = 1$ for some $j \in [1..m]$. Therefore we can join these constraints for each z_{ki} that provides the same set of feasible solutions. The joint users problem $USO(\bar{x}, \bar{y})$ is the system:

$$\begin{aligned}
\sum_{i=1}^m z_{ki} &= 1 & k \in [1..n]. \\
z_{ki} &\leq (\bar{x}_i + \bar{y}_i) & k \in [1..n], i \in [1..m] \\
z_{ki} &\leq 1 - \max_{j=1..m} [c_{ji}^k(\bar{x}_j + \bar{y}_j)] & k \in [1..n], i \in [1..m] \\
z_{ki} &\in \{0, 1\} & k \in [1..n], i \in [1..m]
\end{aligned}$$

that has nm variables z_{ki} , $k \in [1..n]$ and $i \in [1..m]$, and $n(2m + 1)$ linear constraints in these variables.

4.3 The Alternative Location Optimization Problem

Consider now the Alternative Location Set Optimization Problem. Given the proposed locations \bar{x} , this problem consists in selecting the alternative locations y that are preferred by the largest number of users. The corresponding optimization problem, denoted by $ALT(\bar{x})$, can be stated as follows:

$$\begin{aligned}
\text{Maximize} & \quad \sum_{i=1}^m \left[\sum_{k=1}^n w_k z_{ki} \right] y_i \\
\text{Subject to:} & \quad \sum_{i=1}^m y_i = \sum_{i=1}^m \bar{x}_i \\
& \quad z \text{ solves } USO(\bar{x}, y) \\
& \quad y_i \in \{0, 1\} \quad i \in [1..m].
\end{aligned}$$

The objective function counts the number of users that prefer an alternative location point against any proposed location given by the vector set \bar{x} . The first constraint guarantees that the size of the proposed location set and the size of the alternative location set are the same.

If we substitute the second constraint by the linear system (3) we obtain the following optimization problem with linear constraints.

$$\begin{aligned}
& \text{Maximize} && \sum_{i=1}^m \sum_{k=1}^n w_k z_{ki} y_i \\
& \text{Subject to:} && \sum_{i=1}^m y_i = \sum_{i=1}^m \bar{x}_i \\
& && \sum_{i=1}^m z_{ki} = 1 && k \in [1..n] \\
& && z_{ki} - y_i \leq \bar{x}_i && k \in [1..n], i \in [1..m] \\
& && z_{ki} + c_{ji}^k y_j \leq 1 - c_{ji}^k \bar{x}_j && k \in [1..n], i, j \in [1..m] \\
& && y_i, z_{ki} \in \{0, 1\} && k \in [1..n], i \in [1..m]
\end{aligned}$$

However, this problem is not a linear programming problem because the objective includes the solution of $USO(\bar{x}, y)$, as claimed by Dobson and Karmarkar (Dobson and Karmarkar, 1987).

Nevertheless, the alternative location set problem $ALT(\bar{x})$ can be formulated as a linear problem using the sets of users that prefer each single alternative location to the proposed location set.

Proposition 14 *For every m -vector \bar{x} , the Alternative Location Set Optimization Problem $ALT(\bar{x})$ is a linear problem with $n(m+1)$ binary variables and $nm + n + 1$ constraints.*

Proof. Consider the set of users that prefer each single alternative location f_i to the proposed location set given by \bar{x} . This set of users is the set of users “captured” by each single alternative location. Consider the following coefficients defined in a similar way to c_{ij}^k . Let $c_{ki} = 1$ if $d_{ki} < \min\{d_{kj} : \bar{x}_j = 1\}$ and $c_{ki} = 0$ otherwise. If a user at u_k prefers the location f_i to any location in X then, $c_{ki} = 1$; otherwise $c_{ki} = 0$. Then the optimization problem consists in selecting the p alternative location points that jointly capture the maximum number of users.

If a user at u_k prefers the point f_i of the alternative location set Y represented by the vector y , then the variable z_{ki} has to be equal to 1. The condition that states the value of z_{ki} from the coefficient c_{ik} is: $z_{ki} \leq c_{ki}y_i$. Therefore, the alternative location set problem is the following optimization problem:

$$\begin{aligned}
& \text{Maximize} && \sum_{i=1}^m \sum_{k=1}^n w_k z_{ki} \\
& \text{Subject to:} && \sum_{i=1}^m y_i = p \\
& && \sum_{i=1}^m z_{ki} \leq 1 \quad k \in [1..n]. \\
& && z_{ki} - c_{ki}y_i \leq 0 \quad k \in [1..n]; i \in [1..m]. \\
& && y_i, z_{ki} \in \{0, 1\} \quad k \in [1..n]; i \in [1..m].
\end{aligned} \tag{4}$$

The coefficients c_{ik} are constant, since \bar{x} are data for this problem. Then this is an optimization problem with a linear objective function, $n(m + 1)$ binary variables and $1 + n + nm$ linear constraints.

The Alternative Location Set optimization problem can be considered a maximum cover problem because it consists in selecting the p alternative location points that jointly capture the maximum number of users. Procedures for solving these kind of problems are seen in the literature [5].

4.4 The Proposed Location Optimization Problem

The third and last problem concerns Proposed Location Set Optimization. This problem consists in determining the set X of p locations such that the best alternative Y , with the same size, is preferred by the minimum number of users. It is formulated as an optimization problem as follows:

$$\begin{aligned}
& \text{Maximize} && \sum_{i=1}^m \left[\sum_{k=1}^n w_k z_{ki} \right] x_i \\
& \text{Subject to:} && \sum_{i=1}^m x_i = p \\
& && z \text{ solves } USO(x, y) \\
& && y \text{ solves } ALT(x) \\
& && x_i \in \{0, 1\} \quad i \in [1..m].
\end{aligned} \tag{5}$$

This is the problem obtained using the approach proposed by Dobson and Karmarkar. The objective function is non linear in the variables z_{ki} and x_i . In addition, the variables z_{ki} are different for each y and the relation between them is also non linear.

On the other hand, this is the p -Simpson problem that can be formulated as the minimax problem. This problem consists in finding a set X^* such that

$$W^* = W^*(X^*) = \min_{|X|=p} W^*(X) = \min_{|X|=p} \max_{|Y|=p} W(Y \prec X).$$

Its formulation as a linear optimization problem can be done in the same way as for other classical minimax location problems. See for instance the formulation of the p -center problem in Daskin (1995).

Theorem 15 *The p -Simpson problem is a linear optimization problem with $\binom{m}{p}nm + m + 1$ variables and $\binom{m}{p}(nm + n + 1) + 1$ constraints.*

Proof. A minimax problem is formulated as an optimization problem with a linear objective function to minimize, which is the upper bound W of the function that is to be maximized. The problem $W^* = \min_{|X|=p} \max_{|Y|=p} W(Y \prec X)$ in terms of the bound W and the solution X is:

$$W^* = \min\{W : W(Y \prec X) \leq W, \forall Y \subseteq L \text{ with } |Y| = |X| = p\}.$$

The formulation of this problem as a mathematical programming problem is as follows:

$$\begin{aligned} & \text{minimize} && W \\ & \text{subject to:} && |X| = p \\ & && W(Y \prec X) \leq W, \quad \forall Y \in L^p. \end{aligned} \tag{6}$$

This problem has $m + 1$ variables (W, X) and $\binom{m}{p} + 1$ constraints; one for each $Y \subseteq L$ with $|Y| = p$; i.e., $Y \in L^p$. This problem is a linear problem if the constraints $W(Y \prec X) \leq W$ are linear in terms of X for all sets $Y \in L^p$.

The problem of finding a set $X \in L^p$ such that $W(Y \prec X) \leq W$ for all $Y \in L^p$ is similar to the problem $ALT(\bar{x})$ but in that problem we need to find $Y \in L^p$ maximizing $W(Y \prec X)$ for only one set $X \in L^p$. Now we want to find a set $X \in L^p$ minimizing the maximum number of users that prefer another set Y to the set X , for all the sets $Y \in L^p$ at the same time. Note that $W(Y \prec X) = |W(V)| - W(X \preceq Y)$, where $W(X \preceq Y)$ denotes the

number of users that do not prefer Y to X and $W(V)$ is the total number of users. Therefore for solving (6) we maximize $W(X \preceq Y)$ for the $\binom{m}{p}$ possibilities for the set Y at the same time.

There are $\binom{m}{p}$ possible selections for $Y \in L^p$. Let j be an index $j \in J = \{1, 2, \dots, \binom{m}{p}\} = [1.. \binom{m}{p}]$ representing the possible selections for a set $Y \in L^p$. Let Y_j denote the corresponding set.

Let c_{ki}^j be the coefficients that determine the users that do not prefer a point of Y_j to the facility point f_i . They are given by: $c_{ki}^j = 1$ if $d(u_k, f_i) \leq d(u_k, Y_j)$ and $c_{ki}^j = 0$ otherwise. Let the variable z_{ki}^j be equal to 1 if the users at u_k select the facility location f_i in X when comparing Y_j and X ; 0 otherwise. Assume that if the users at u_k have several points of X at the same distance; all those users select only one of these points. The conditions that state the values of z_{ki}^j are $z_{ki}^j \leq c_{ki}^j x_i, \forall i, k, j$, and $\sum_{i=1}^m z_{ki}^j \leq 1, \forall k, j$.

The number of users that do not select an alternative location of Y_j , when comparing it with X , is $W(X \preceq Y_j) = \sum_{i=1}^m \sum_{k=1}^n w_k z_{ki}^j$. Therefore, the global problem is:

$$\begin{aligned}
& \text{Minimize} && w \\
& \text{Subject to:} && \sum_{i=1}^m x_i = p \\
& && W(V) - \sum_{i=1}^m \sum_{k=1}^n w_k z_{ki}^j \leq w \quad j \in J. \\
& && \sum_{i=1}^m z_{ki}^j \leq 1 \quad k \in [1..n]; j \in J. \\
& && z_{ki}^j - c_{ki}^j x_i \leq 0 \quad k \in [1..n]; i \in [1..m]; j \in J. \\
& && x_i, z_{ki}^j \in \{0, 1\} \quad k \in [1..n]; i \in [1..m]; j \in J. \\
& && w \geq 0
\end{aligned} \tag{7}$$

There are the variable w , the m variables x_i and $nm \binom{m}{p}$ variables z_{ki}^j ; i.e., $\binom{m}{p} nm + m + 1$ variables. Therefore, this is a linear problem with $\binom{m}{p} nm + m + 1$ binary variables and $\binom{m}{p} (nm + n + 1) + 1$ constraints. This concludes the proof of the theorem.

Unfortunately, this problem is practically intractable by standard linear programming techniques. The appropriate approaches to deal with this linear formulation are the column and row generation procedures that allow only those variables and constraints that are relevant for the solutions near the optimum to be included.

5 Solution Approach

The core solution procedure for the multiple voting location problems is an algorithm to obtain a p -Simpson set X^* by solving the minimax problem:

$$W^* = W^*(X^*) = \min_{|X|=p} W^*(X) = \min_{|X|=p} \max_{|Y|=p} W(Y \prec X). \quad (8)$$

In order to solve the p -Condorcet set problem, find a p -Simpson set X^* and then if $W^* \leq |U|/2$ then X^* is a p -Condorcet set, otherwise there is no p -Condorcet solution. The plural and security problems are solved by replacing the formulas corresponding to $W(Y \prec X)$ with those corresponding to $W(Y \prec X) - W(X \prec Y)$. The other extensions of the problem are solved by introducing the corresponding parameters, γ for the rejection majority and α for the indifference threshold. The tolerant versions are solved by a dichotomic search in the entries of the distance matrix $D = [d_{ki} = d(u_k, f_i)]$ for the value τ ; the minimum tolerance α such that a p -Condorcet with tolerance α exists. Then a τ -Condorcet Set is a Tolerant Condorcet Set. Since the distance matrix has mn entries, $O(\log n + \log m)$ p -Simpson problems have to be solved to find the p -Tolerant Condorcet location problem. Analogously, the Tolerant Plural set can be obtained by solving $O(\log n + \log m)$ p -Security problems.

In order to solve the minimax problem (8) note that, given $X^* \in L^p$, if no location set $X \in L^p$ such that $W^*(X) < W^*(X^*)$ exists, then X^* is the solution to the problem. Moreover, given any upper bound W' of the objective function and a selected family F of subsets of L^p (i.e., $F \subseteq L^p$), if no set $X \in L^p$ such that $W(Y \prec X) < W', \forall Y \in F$, exists then the problem of finding a set X such that $W^*(X) < W'$ has no solution. The procedure tries to use a short family F of good candidates to conclude that the upper bound provided by one of them, say $W' = W^*(X')$, cannot be improved and then X' is optimal. To determine if a set $X \in L^p$ such that

$$W(Y \prec X) < W', \forall Y \in F, \quad (9)$$

for a family F exists, we can solve the following minimax problem similar to (8) but using $Y \in F$ instead of $Y \in L^p$.

$$W_F^* = W_F^*(X_F^*) = \min_{|X|=p} W_F^*(X) = \min_{|X|=p} \max_{Y \in F} W(Y \prec X). \quad (10)$$

This problem is also a linear LP problem very similar to (7) that is formulated as follows.

$$\begin{aligned} & \text{Minimize} && w \\ & \text{Subject to:} && \sum_{i=1}^m x_i = p \\ & && W(V) - \sum_{i=1}^m \sum_{k=1}^n w_k z_{ki}^j \leq w \quad j \in F. \\ & && \sum_{i=1}^m z_{ki}^j \leq 1 \quad k \in [1..n]; j \in F. \\ & && z_{ki}^j - c_{ki}^j x_i \leq 0 \quad k \in [1..n]; i \in [1..m]; j \in F. \\ & && x_i, z_{ki}^j \in \{0, 1\} \quad k \in [1..n]; i \in [1..m]; j \in F. \\ & && w \geq 0 \end{aligned} \quad (11)$$

This problem has $|F|nm + m + 1$ variables and $|F|(nm + n + 1) + 1$ constraints. Therefore the size of the problem depends on the size of family F .

The proposed solution procedure is based on the use of a family F composed by good candidates that provide good bounds and good solutions that are iteratively improved while possible. Several instances of the problem (10) are solved. If the size of the family F is kept low and it consists of good candidates for the problem then the efficiency of the procedure is improved. Each candidate included in F gives information to reject other candidates.

- Each candidate location set $X \in L^p$ included in F gives an upper bound $W^*(X)$ for W^* . The value $W^*(X)$ is obtained by solving the problem $\max\{W(Y \prec X) : Y \in L^p\}$ as a linear problem (4).
- Each alternative location set $Y \in L^p$ included in F also gives a constraint for the optimal solution X . The constraint is $W(Y \prec X^*) < W'$ where W' is any upper bound of W^* .

The method successively searches for a good candidate that is tested and included in the family F . The value $W^*(X) = \max\{W(Y \prec X) : Y \in L^p\}$

for each selected candidate is obtained and the best candidate tested X' is kept. The candidates Y such that $W(X \prec Y) \geq W^*(X')$ are discarded as possible candidates. For every given family F , if the solution of the problem (10) does not satisfy $W < W^*(X')$ then X' is a p -Simpson set. Otherwise, a new good candidate to be included in F is obtained. This new candidate Y is that which maximizes $W(Y \prec X)$ for the last candidate X . Each new candidate Y added to the family F means that the nm variables and $nm + m + 1$ constraints corresponding to Y are added to the problem (11).

The algorithm to solve the p -Simpson problem has the following steps:

Algorithm pS

Step 1 Initialization.

- 1.1 Let C be the list of possible candidates. Initially C includes all the location sets with p points; i.e., $C = L^p = \{X \subseteq L : |X| = p\}$.
- 1.2 Let F be the selected family of good candidates. Initially F is empty; $F = \emptyset$.
- 1.3 Let $W' = |U|$ be the first bound for the objective function.
- 1.4 Select any set X_1 as the first candidate for F .
- 1.5 Let $i = 0$.

Step 2 Iterations. Repeat, until C is empty, the steps:

- 2.1 Do $i = i + 1$. Delete X_i from C and add it to F .
- 2.2 Compute the bound $W_i = W^*(X_i) = \max\{W(Y \prec X_i) : Y \in L^p\}$.
- 2.3 If $W_i < W'$ then $W' = W_i$ and $X' = X_i$.
- 2.4 Take $Y_i = X_i$ and delete from C all the candidates X such that $W(Y_i \prec X) \geq W'$. If $W_i = W'$ then, for each $j < i$, delete from C all the candidates X such that $W(Y_j \prec X) \geq W'$ for the candidate $X_j \in F$.
- 2.5 If C is nonempty then select from C the set X with the maximum value for $W(X \prec Y_i)$ as the new candidate X_{i+1} .

Note that, at step 2.2., the bound is obtained by solving linear problem (4). At step 2.4, the constraint $W(Y_i \prec X) < W'$ is introduced in (11) when the set Y_i is added to the family F . If the bound W' decreases at step 2.3, then the constraints $W(Y_j \prec X) < W'$ for $j = 1, \dots, i - 1$, (that were already introduced in the problem) are now tightened. At step 2.4, problem (1) is used to reject the candidate sets that violate the new constraint or the tightened ones. Finally, at step 2.5, the new candidate is the optimal solution to problem (4).

Procedure efficiency can be improved at several steps. The number of candidate tested is reduced by selecting a good first candidate X_1 at step 1.4. (for instance, the p -median of U in L ; see [10]). Some comparisons are also avoided if the bound $W_i = W^*(X_i)$ at step 2.2 is obtained in two steps:

$$\begin{aligned} W_i^1 &= \max_{Y \in F \cup C} W(Y \prec X_i). \\ W_i^2 &= \max_{Y \notin F \cup C} W(Y \prec X_i). \end{aligned}$$

Note that, $F \cup C$ will be small and consist of good candidates where the maximum of $W^*(X_i)$ is likely to be reached. Then to test if $W_i \geq W'$, first test if $W_i^1 \geq W'$. If this condition is met then it is not necessary to compute W_i^2 and numerous computations are avoided. Finally, if the bound W' does not decrease at step 2.3 ($W_i \geq W'$) then the only candidates deleted from C are the sets X such that $W(X_i \prec X) \geq W'$, thus avoiding comparing them with all the sets $X \in F$ at step 4.

6 Running time analysis

It is well known that the usual algorithms to solve linear programs, like the Simplex method, have worst-case exponential running time. However their practical performance is usually good and there are commercial implementations that can solve large size problems efficiently. These implementations, like CPLEX combined with Branch and Bound and other techniques, can solve integer linear programs of a moderate size. The number of constraints in some of the linear problems that appear in this approach is also exponentially growing (unless p is a constant). However, during the execution of the algorithm the number of constraints depends on the size of the family F that is kept low.

A straightforward enumeration of the solution space can be executed to find the optimal solution of only small size problems. The size of the solution space for the proposed location and the alternative location problems is $|L^p| = \binom{m}{p}$. This size also grows exponentially for general p , but it is polynomial for every fixed constant p . Moreover, the evaluation of each solution for the proposed location problem needs to solve another problem (the alternative location problem) that also has the same size; $\binom{m}{p}$ that is $O(m^p)$. Therefore, solving the proposed location problem by a straightforward enumeration of the solution space means to perform an $O(m^p)$ enumeration $O(m^p)$ times. The users problem has to be solved at each step of this last enumeration, and this can be done in $O(pn)$ time; the time needed to compare the proposed and alternative locations, for every user.

The gain with our approach is a great reduction in the number of candidate locations that are evaluated. This reduction is for the number of evaluations of proposed locations, and for the number of evaluations of alternative locations for each proposed location. The reduction is achieved by using an intelligent strategy to select the good candidates included in F , that are the only solutions evaluated and compared with other solutions to be discarded. A straightforward implementation of the algorithm was used to experimentally test the number of candidates evaluated and alternative locations compared in the execution of the algorithm with instances of several size. The results are used to support our claims.

We randomly generated instances on a (50×50) grid graph $G_{50,50}$ where each edge has a length equal to 1. The instances were generated as follows. Given the number m of user locations and the number n of possible location points, we randomly selected m vertices to be user vertices and n vertices to be possible locations. The number of users at each user vertex was randomly selected in the interval $[1, \bar{w}]$ (we chose $\bar{w} = 20$ for our experiments).

We chose $m \in \{20, 30, 40, 50\}$ and $n \in \{m, 1.5m, 2m\}$ and two values for p (4 and 5). We generate 10 instances for each combination of values for n , m and p . Tables 1 and 2 show average values for the number of candidates evaluated and the number of comparisons made in solving each set of instances. Note that the complete enumeration approach would require the evaluation of $\binom{m}{p}$ candidates by performing $\binom{m}{p}$ comparisons; i.e. a total of $\binom{m}{p}^2$ comparisons.

m	n	p	Size	Evaluations		Comparisons	
20	20	4	4845	79.1	(1.63%)	147715.2	(0.63%)
20	30	4	4845	90.5	(1.87%)	176776.2	(0.75%)
20	40	4	4845	75.2	(1.55%)	145043.6	(0.62%)
30	30	4	27405	218.1	(0.80%)	3470803.6	(0.46%)
30	45	4	27405	246.8	(0.90%)	3955609.3	(0.53%)
30	60	4	27405	253.2	(0.92%)	4203465.4	(0.56%)
40	40	4	91390	277.8	(0.30%)	17488656.9	(0.21%)
40	60	4	91390	343.0	(0.38%)	22298038.3	(0.27%)
40	80	4	91390	235.8	(0.26%)	14950280.5	(0.18%)
50	50	4	230300	508.3	(0.22%)	92316114.0	(0.17%)
50	75	4	230300	508.3	(0.22%)	93029516.0	(0.18%)
50	100	4	230300	214.0	(0.09%)	35282350.2	(0.07%)

Table 1. Average number of evaluations and comparisons for $p = 4$.

m	n	p	Size	Evaluations		Comparisons	
20	20	5	15504	96.5	(0.62%)	297489.2	(0.12%)
20	30	5	15504	157.8	(1.02%)	522922.6	(0.22%)
20	40	5	15504	143.3	(0.92%)	449244.7	(0.18%)
30	30	5	142506	388.1	(0.27%)	22663548.3	(0.11%)
30	45	5	142506	595.4	(0.42%)	32441217.5	(0.16%)
30	60	5	142506	639.8	(0.45%)	41353591.7	(0.20%)
40	40	5	658008	940.6	(0.14%)	335547489.1	(0.08%)
40	60	5	658008	1158.2	(0.17%)	424947993.3	(0.10%)
40	80	5	658008	953.3	(0.14%)	329778733.2	(0.08%)
50	50	5	2118760	2513.3	(0.12%)	120649812.2	(0.01%)
50	75	5	2118760	970.0	(0.05%)	860498115.1	(0.02%)
50	100	5	2118760	1573.7	(0.07%)	764394573.3	(0.02%)

Table 2. Average number of evaluations and comparisons for $p = 5$.

The numerical results show that the number of evaluations and comparisons is a very low percentage of the corresponding numbers in an exhaustive enumeration. This percentage is significantly influenced by the number of user points, m . It decreases with the number m of possible location points and with the number p of points to be located.

7 Conclusions

Few modifications or extensions of the solution approach proposed in this paper have to be included for solving the extensions of the main problem; the p -Simpson problem. The p -Condorcet is found by solving the p -Simpson problem; if the value $W^*(X^*)$ for its optimal solution X^* is not greater than one half of the users ($|U|/2$) the X^* is a p -Condorcet set, otherwise the p -Condorcet solution does not exist. The plural and security problems are solved by replacing the function that computes $W(Y \prec X)$ for another function computing $W(Y \prec X) - W(X \prec Y)$. The parameter α can be introduced in sentences that compare the distances between a user point and two possible location points. The parameter γ can be introduced at the final step of solution procedure by comparing with $\gamma|U|$ instead of with $|U|/2$. The Tolerant problems are solved by a dichotomic search in the corresponding distance matrix entries for the tolerance τ ; this means applying the pS algorithms for different values of α . Note that in this case, the good candidates for a value of α would also be good candidates for another value.

A similar approach can be applied to an infinite set of location points if we previously find a finite dominant set, as is usual in network location problems (see Pérez Brito and Moreno Pérez, 2000). The approach can be applied if the possible solutions are extensive locations like paths, cycles or trees by finding a way to implicitly or explicitly enumerate them. For instance, for the location of a cycle or a path with bounded length l to serve a set of users we need to consider the families of the set of edges that compound a cycle or a path instead of the location space L^p .

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