The Generalized p-Centdian on Network

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Since the median approach is based on averaging, it often provides solutions where remote and low-population density areas are discriminated against in terms of accessibility to public facilities, as compared with centrally situated and high-population density areas. On the other hand, locating a facility at the centre may cause a large increase in total distance, thus generating a substantial loss in spacial efficiency. This has led to a search for some compromise solution concept. Halpern has introduced the λ-Centdian as a parametric solution concept based on the bicriteria center median model. He has modeled the corresponding trade-offs with a convex combination of the unweighted center and weighted median objectives. In this paper we study the λ-Centdian like a convex combination of the weighted center and weighted median objectives we called it generalized Centdian problem.

Location, Centdian, Networks.

68Q20, 68R99, 05C85, 90C35.

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Introduction

Location analysis deals with the location of facilities taking into account their effect on the population. In a typical problem on a network there is a set of customers located at some points on the network. The goal is to locate new facilities on the network in order to minimize the cost of serving the customers. In most location models this cost is assumed to be

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a monotone nondecreasing of the distances between the customers and the servers. The most fundamental and common problems studied in network location theory are the median and the center models. The underlying assumption in both models is that the servers are identical and uncapacitated, and as a result each customer will be served by the nearest server. The median is suitable for locating a facility providing a routine service, by means of minimizing the average distances of users to it. The center is appropriate for emergency services where the objective is to have the furthest users as close as possible to the center.

In many real world problems the objective is a mixture of different, possibly adverse objectives, for example, in locating a fire station one may want to minimize the travelling time to the farthest potential source of a call for service as well as one may try to locate as close as possible to the heavily populated areas. The problem is therefore to minimize both objective functions. Such goal may be mathematically expressed by minimizing a new objective function that is a convex combination of the objective functions of the center and median problems. This multi-objective approach for locating a facility on a network was introduced by Halpern (1976), who coined the term **Cent-dian** for the points which minimize the convex combinations of the center and median objective functions.

Ever since the seminal paper by Hakimi (1964), a thread running through network location theory is the identification of a finite subset of the network that necessarily contains an optimal solution for all the instances of a particular location problem. See Hooker et al. (1991) for a review on this kind of results. Since Hakimi (1964), it is known that the set of vertices is a finite dominating set for the p-median problem. The set of vertices and local centers (points, in the interior of the edges, that are equidistant and balanced with respect to vertices) is a finite dominating set for the p-center problem; e.g. see Moreno (1985). From Pérez Brito, Moreno Pérez and Rodríguez Martín (1997), it is known that the set of vertices, local centers and extreme points of the network is a finite dominating set for the p-Centdian problem.

In this paper we focus in the generalized p-Centdian problem on a network, it means that the weights w_i appear in the maximum distance and in the average distance, the weights are not there necessarily equal for both functions. Note that in the Halpern model the weights w_i do not appear in the formula for the maximum distance. In typical applications, the weight represents the number of custumers located at the corresponding vertex. These different weights appear when the facility points have to serve a set of customer that demand for a regular service and also for an emergency service from the same points.

The main contribution of the paper is the identification of a set of points of polynomial size, which is guaranteed to contain an optimal solution. Also we show the first exact algorithm for the generalized p-Centdian problem on a network. If the underlying network is a tree there is a polynomial algorithm due to Tamir, Pérez-Brito and Moreno Pérez (1998).

Next section provides the basic definitions and notation for the formulation of the generalized p-facility Centdian problem on network that is derived from the classical p-center and p-median problems. Section 3 includes the sets of special points that we will need for the dominating set. The following section reviews the 1-λ-Centdian statement, adapting the main properties of Halpern model (Halpern (1978) and (1980)) to the new model with weight in the center function. Section 5 shows the finite dominating set for the generalized p-Centdian problem on a network. Section 6 includes an exact algorithm for every p. The paper ends wind a short conclusion section.

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Formulation of the problem.

Let $N=(V,E)$ be an undirected network with a node set $V=\{v_1,...,v_n\}$ and an edge set $E=\{e_1,...,e_m\}$. Each edge has a positive lenght and is assumed to be rectifiable. We refer to interior points on an edge by their distance along the edge from the two nodes of the edge. An interior point on an edge divides it into two subedges whose lengths are respectively the distance from such a point to the vertices. Let $P(N)$ denote the continuum set of points on the edges of N. The edge lenghts induce a distance function $d(.,.)$ on $P(N)$. A path between two vertices i and j is a minimal sequence of edges of N joining i and j. A path betweeen two points x and y is a path between two vertices plus two subedges joining them with the points. The length of a path is the sum of leghts of its edges and subedges. For any pair of points $x,y\in P(N)$, let $d(x,y)$ be the minimum lenght of the paths between them. Also, for any subset

$X \subset P(N)$ its distance to a vertex $v_i \in V$ is given by:

$$d(X, v_i) = \min_{x \in X} d(x, v_i).$$

$P(N)$ is a metric space with respect to the above distance function.

Suppose that each node $v_i \in V$ is associated with a pair of nonnegative weights (w_i, w_i'), representing in typical applications the number of customers located at the corresponding vertex. Using the above notation we now define the p-center, the p-median and the p-Centdian problems. (Note that the node set V is identified as the set of customers in these problems).

The median problem consists of determining the locations of the set of facilities that minimizes average travel time to or from the facilities, for the population of their users. For a given value of $p \geq 1$, the so called p-median problem is to establish p facilities in p potential locations and to supply each user from the established facilities such that the demands of all users are met and the total costs thereby incurred are minimized. The p-center problem is to open p facilities and to assign each user to exactly one of them such that the maximum distance from any open facility to any of the users assigned to it is a minimum.

Given the network $N = (V, E)$ and the set U of vertices where are the users that have to be served, the w_i weighted p-median problem is to find the set $X^* \subset P(N)$, subject to $|X^*| = p$, that minimizes the objective function:

$$f_m(U; X) = \sum_{v_i \in U} w_i \, d(X, v_i).$$

The w_i' weighted p-center problem is to find the set $X^* \subset P(N)$, subject to $|X^*| = p$, that minimizes the objective function:

$$f_c(U; X) = \max_{v_i \in U} w_i' \, d(X, v_i).$$

Total -or average- distance minimization tends to favor users who are clustered in population centers to the detriment of users who are spatially dispersed. Discrimination of this kind with regard to accessibility may have a severe impact on remote users in the case of an emergency service (ambulances, fire brigades, police cars,...). As a result, the decision maker may want to consider a criterion focusing more on users who get poorly served.

For a given λ, $0 \leq \lambda \leq 1$, the **generalized λ-Centdian** problem is to find the location that minimizes the objective function defined by: $f_\lambda = \lambda \cdot f_c + (1-\lambda) \cdot f_m$, where f_c and f_m are the objective functions of the center and median problems, respectively. The value of λ reflects the weight attributed to the center function with respect to the median function. When $\lambda = 0$, the generalized λ-Centdian problem is the median problem and when $\lambda = 1$, it is the center problem. For $0 < \lambda < 1$, it can be viewed as a location problem where both efficiency and equity criteria are taken into account; the generalized λ-Centdian is also a location that minimizes a linear combination of the average and maximum distances to the user vertices.

Given the network N and the set of user vertices U, the single facility generalized λ-Centdian problem consists in finding the point $x^* \in P(N)$ such that:

$$f_\lambda(U;\{x^*\}) = \min_{x \in P(N)} f_\lambda(U;\{x\}).$$

The **generalized p-λ-Centdian problem** consists in finding the set $X^* \subset P(N)$, subject to $|X^*| = p$, that minimizes the objective function

$$f_\lambda(U;X) = \lambda \cdot f_c(U;X) + (1-\lambda) \cdot f_m(U;X).$$

i.e., such that

$$f_\lambda(U;X^*) \leq f_\lambda(U;X), \forall X \subset P(N), \text{ with } |X| = p.$$

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Special Sets of points.

The following sets of interior points of the edges of the networks are used in the finite dominating sets for the p-facility location problems on networks (see figure 1).

- A point $x \in P(N)$ is a **Local Center** with range r associated to user vertices v_k, $v_l \in U$, (we denote $x \in B_c(r; v_k, v_l)$) if x is an interior point of an edge $[i,j]$ such that:

1. $r = w'_k\, d(x, v_k) = w'_k\, (l(x, i) + d(i, v_k)) < w'_k\, (l(x, j) + d(j, v_k))$ and
2. $r = w'_l\, d(x, v_l) = w'_l\, (l(x, j) + d(j, v_l)) < w'_l\, (l(x, i) + d(i, v_l))$.

For each value of $r \geq 0$, let us define:

$$LC(r) = \bigcup_{v_k, v_l \in U} B_c(r; v_k, v_l)$$

- A point $x \in P(N)$ is a **Pendant Point** with range r associated to user vertices v_k, $v_l \in U$, (we denote $x \in B_p(r; v_k, v_l)$) if x is an interior point of an edge $[i, j]$ such that:

  1. $r = w'_k\, (l(x, i) + d(i, v_k)) = w'_l\, (l(x, i) + d(i, v_l))$ with $w'_k \neq w'_l$, or
  2. $r = w'_k\, (l(x, j) + d(j, v_k)) = w'_l\, (l(x, j) + d(j, v_l))$ with $w'_k \neq w'_l$.

  For each value of $r \geq 0$, let us define:

  $$PP(r) = \bigcup_{v_k, v_l \in U} B_p(r; v_k, v_l)$$

- A point $x \in P(N)$ is a **Bottleneck Point** with range r associated to user vertex $v_k \in U$, (we denote $x \in B_a(r; v_k)$) if x is an interior point of an edge $[i, j]$ such that:

  1. $r = w'_k\, d(x, v_k) = w'_k\, (l(x, i) + d(i, v_k))$, and
  2. $r = w'_k\, d(x, v_k) = w'_k\, (l(x, j) + d(j, v_k))$.

  For each value of $r \geq 0$, let us define:

  $$BP(r) = \bigcup_{v_k \in U} B_a(r; v_k).$$

- A point $x \in P(N)$ is an **Extreme Point** with range r associated to user vertex $v_k \in U$, (we denote $x \in E_p(r; v_k)$) if x is an interior point of an edge $[i, j]$ such that:

  1. $r = w'_k\, d(x, v_k) = w'_k\, (l(x, i) + d(i, v_k))$ or
  2. $r = w'_k\, d(x, v_k) = w'_k\, (l(x, j) + d(j, v_k))$.

For each value of $r \geq 0$, let us define:

$$EP(r) = \bigcup_{v_i \in U} E_p(r; v_i).$$

Note that, for each value $r \geq 0$ and for each user vertex $v_i \in U$, we obtain $B_c(r; v_i, v_j) \subseteq E_p(r; v_i)$, $B_p(r; v_i, v_j) \subseteq E_p(r; v_i)$ and $B_a(r; v_i) \subseteq E_p(r; v_i)$. Therefore, for each value $r \geq 0$, we have $LC(r) \cup PP(r) \cup BP(r) \subseteq EP(r)$.

Let also be, for each value of $r \geq 0$ and vertex $v_i \in U$,

$$VV(r; v_i) = \{v_j \in V : r = w'_i d(v_i, v_j)\}$$

and, for each value of $r \geq 0$, let us define:

$$VV(r) = \bigcup_{v_i \in U} VV(r; v_i).$$

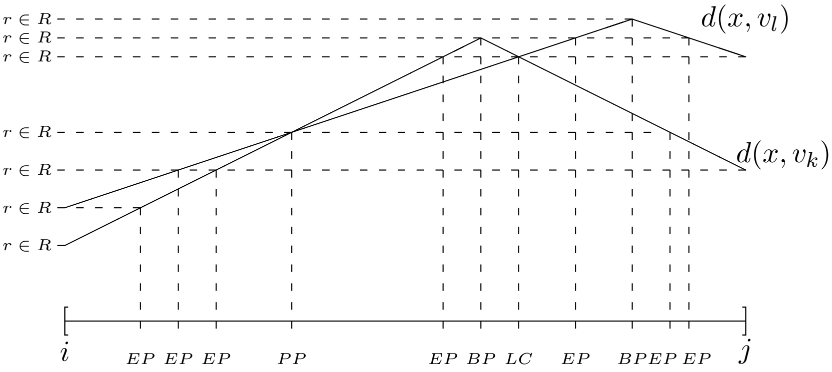

Figure 1: Canonical distances and special points on an edge.

The union of the set of local center ranges, the set of bottleneck point ranges, the set of pendant point ranges and the set of weighted distances between the vertices and the user vertices are given by

$$\begin{aligned} R \quad = \quad & \{r : LC(r) \neq \emptyset\} \cup \{r : PP(r) \neq \emptyset\} \cup \\ & \{r : BP(r) \neq \emptyset\} \cup \{r : VV(r) \neq \emptyset\}, \end{aligned}$$

which will denote the **canonical distances set**. The set of extreme points with range in this set are the canonical extreme points (see figure 1).

Lemma 0.1.

$$|R| = O(mn^2)$$

Proof: There is at most a bottleneck point in each edge associated to every user vertex. Then the distance from an interior point on an edge $[i, j]$ to an user vertex v_k out of the edge is increassing from i to the bottleneck point and decreasing from it to j. Therefore there are at most a local center and two pendant points in each edge associated to every pair of user vertices. Thus the number of different ranges of local center, bottleneck and pendant points is $O(mn^2)$. The number of weighted distances between the vertices and the user vertices is $O(mn^2)$. Therefore $|R| = O(mn^2)$. □

The set of associated local centers, pendant points, bottleneck points and canonical extreme points are:

$$LC = \bigcup_{r \in R} LC(r)$$

$$PP = \bigcup_{r \in R} PP(r)$$

$$BP = \bigcup_{r \in R} BP(r)$$

$$EP = \bigcup_{r \in R} EP(r).$$

Lemma 0.2.

$$|EP| = O(mn^3)$$

Proof: For each $r \in R$ there are at most two extreme points in each edge associated to any vertex, then $|EP| = O(mn^3)$. □

Note that, by definition of R and EP, we also have:

$$V \cup LC \cup PP \cup BP \subseteq EP.$$

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The Generalized 1-λ-Centdian.

The goal is to locate a new facility point (server) on the network in order to minimize

$$f_\lambda(U;\{x^*\}) = \min_{x\in P(N)} \{f_\lambda(U;\{x\})\}\,.$$

In this section we review the problem statement, about which references are scarse; this be done by adapting the main properties of Halpern model (Halpern (1978)) to the new model with weights in the center function. In this section, the objective function $f_\lambda(U;\{x\})$ is denoted by $f_\lambda(x)$

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Properties of the Generalized 1-λ-Centdian.

Let $e=[v_i,v_j]$ be an edge of E, and Q_e the breakpoints set of $f_c(x)$ in e plus v_i and v_j.

The following result is due to the properties of $f_c(x)$ and $f_m(x)$.

Proposition 0.1. *Given λ, $0\le\lambda\le 1$, the function of x:*

$f_\lambda(x)=\lambda f_c(x)+(1-\lambda)f_m(x)$, $x\in e=[v_i,\, v_j]$, *is a function of x continuous and piecewise linear function, with a finite number of breakpoints over e, all belonging to $Q_e\subset e\cap(LC\cup PP\cup BP\cup V)$.*

Proof: $f_\lambda(x)$ is a piecewise linear continuous function because it is the upper envelope of two piecewise linear continuous functions: $\lambda f_c(x)$ and $(1-\lambda)f_m(x)$. Each of these functions has all the breakpoints belonging to $Q_e\subset e\cap(V\cup LC\cup PP\cup BP)$. □

From the above proposition it is easy to deduce that for a given λ, $0\le\lambda\le 1$, and two consecutive points x and y belonging to Q_e, if $f_\lambda(x)=f_\lambda(y)$, then the function f_λ have to be constant on $[x,\, y]$. Thus, all points on $[x,\, y]$ belong to Q_e (Hansen et.al. (1991)).

Proposition 0.2. *The set $LC\cup PP\cup BP\cup V$ is a finite dominating set for the 1-λ-Centdian problem.*

Proof: It follows from the proposition above. □

Let $f_\lambda = \; min\{f_\lambda(x) : x \in P(N)\}$. For a given λ, f_λ denotes the value of 1-λ-Centdian function on a λ-Centdian point.

Using the above proposition, we could write f_λ as follows:

$$f_\lambda = \min\{\lambda\, f_c(q) + (1-\lambda)\, f_m(q) : q \in \bigcup_{e\in E} Q_e\}.$$

Since Q_e is a finite set, we have the next proposition:

Proposition 0.3. *The function* f_λ *(as function of* λ*) is continuous, piecewise and concave with* $0 \le \lambda \le 1$.

Proof: For each $q \in \bigcup_{e\in E} Q_e$ the expression $\lambda\, f_c(q) + (1-\lambda)\, f_m(q)$ represent a line segment, which is continuous and lineal. The minimum of all these segment lines is continuous, piecewise and concave. □

Definition 0.1. A solution x **dominates** another solution y over the interval $[\lambda', \lambda''] \subseteq [0,1]$, if $f_\lambda(x) \le f_\lambda(y)$ for all λ such that $\lambda' \le \lambda \le \lambda''$.

Definition 0.2. A solution x is **dominant** over $[\lambda',\ \lambda'']$, if it dominates all the solutions over that interval.

Definition 0.3. A solution y is **uniformly dominating** by a subset of solution $J \subset P(N)$, if for every $\lambda \in [0,\ 1]$ there exists $x \in J$ such that $f_\lambda(y) \ge f_\lambda(x)$; such a case will be denoted by $y \dashv J$.

Proposition 0.4. *There are two finite sets,* $\{\lambda_i : i = 0, ..., r\}$ *of values on* $[0,1]$ *and* $\{x_i : i = 1, ..., r\}$ *of points of* $P(N)$*, which have the next properties:*

(i) $\lambda_i < \lambda_{i+1}$ *for all* i*;* $\lambda_0 = 0$, $\lambda_r = 1$.

(ii) $x_1 = x_m$ *(Median),* $x_r = x_c$ *(Center).*

(iii) $f_\lambda = f_\lambda(x_i)$ *for all* λ *such that* $\lambda_{i-1} \le \lambda \le \lambda_i$, $i = 1, ..., r$.

Proof: Let $\{\lambda_i : i = 0, ..., r\}$ be the breakpoints of f_λ, from the proposition 4.2, if λ_{i-1} and λ_i are two adjacent points of discontinuity in the derivative of f_λ, then there are some $x_i \in P(N)$ such that x_i is dominant on $[\lambda_{i-1},\ \lambda_i]$ and $f_\lambda = f_\lambda(x_i)$ for all $\lambda_{i-1} \le \lambda \le \lambda_i$. □

The following proposition shows how to find the 1-λ-Centdian in a general network.

Proposition 0.5. *The* 1-λ-*Centdian in a network, for any* $\lambda \in [0,\ 1]$, *is located on a path connecting the Center and the Median of the network.*

Proof: From the proposition 4.4, the path connecting the Center and the Median of the network is the path of such points x_i, where for each λ_i the points of $[x_i, x_{i+1}]$ are $\lambda_i - CD$. □

Proposition 0.6. *If* $x^*(\lambda)$ *is* 1-λ-*Centdian for a given* λ *then*$f_c(x^*(\lambda))$ *is a function nonincreasing of* λ, *while* $f_m(x^*(\lambda))$ *is a function nodecreasing of* λ.

Proof: Some of the points $\{x_i : i = 1, ..., r\}$ from the proposition 4.4 are λ-Centdian for all $\lambda \in [0, 1]$ and also satisfy that $f_c(x_i) \geq f_c(x_{i+1})$ and $f_m(x_i) \leq f_m(x_{i+1})$, $i = 1, ..., r-1$. Moreover, $x^*(0) = x_m = x_1$ y $x^*(1) = x_c = x_r$, which proves the property. □

In order to resolve the 1-λ-Centdian in a network there are two algorithms with the same complexity $O(mn \log mn)$, Halpern (1978) and Hansen et. al. (1991). In both cases the algorithms find a set of points λ-Centdian, for all $\lambda \in [0, 1]$.

The Halpern method, uses upper bound for the minimal value of f_λ, while the Hansen et. al., resolve the problem using geometric concepts. The last one is easier to understand.

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The Generalized p-λ-Centdian.

The objective of the generalized p-λ-Centdian is to find p points $X = (x_1^*, x_2^*, ..., x_p^*) \in P(N)^p$ such that:

$$f_\lambda(U; (x_1^*, x_2^*, ..., x_p^*)) = \min_{X \in P(N)^p} \{f_\lambda(U; X)\}.$$

Next theorem provides the finite dominating set for the generalized p-centdian problem. This result, the consequent propositions, the algorithm to solve the problem and the complexity proved in section 6 are very similar to that in Pérez-Brito et al. (1997) for the unweighted case. The main idea is that, if some facility point x_k in a solution X is not in the proposed finite dominating set D then the set X can be modified without increasing the objective function until all the points in X are also in D. We show how to move one of its points, or some of them (at the same time) an inversally weighted amount until special points are reached.

The proof of the theorem, that is the main result of the paper, is almost identical to that in Pérez-Brito et al. (1997). Here, in the weighted case, either when (i) (see $2a$ in Case 2) the facility point x_k interior to an edge $[i, j]$ varies by a small amount ξ toward the vertex j or toward the vertex i, or when (ii) (see $2b$ in Case 2) all the points x_k, $k \in K$ vary at the same time by an amount ξ/w'_k in the appropiated direction, it is possible to reach a pendant point. The basic difference with the aforementioned paper is that in the unweighted case such a point does not exist. In addition there are some algebraic differences which are due to considering weights in the center function: for example in Case 2 (2a), the derivative of λf_c function is $\pm\lambda w'_k$ instead of $\pm\lambda$ that is the derivative in the unweighted case. The proof is large and exhaustive and it is included here in order to allow to the reader to understand all the details of the proving technique.

Theorem 0.1. *The set EP of canonical extreme points of the network associated to user vertices is a finite dominating set for the generalized p-λ-Centdian problem.*

Proof: Let every candidate solution be given by a vector $X \in P(N)^p$; i.e., a selection of p facility points $X = (x_1, x_2, ..., x_p)$ with $x_k \in P(N)$, for $k = 1, ..., p$. Every user vertex $v_i \in U$ is assigned to its closest component of $X \in P(N)^p$, then the result is a series of sets $U(X) = (U_1(X), U_2(X), ..., U_p(X))$ defined by:

$$U_k(X) = \{v_i \in U : w'_i d(x_k, v_i) = \min_{x \in X} w'_i d(x, v_i)\}.$$

They constitute a partition of the set of user vertices U only if there is not a tie; otherwise an optimal partition can be established by solving the ties arbitrarily and then the partition is not unique. The assignment given by the sets $U^*(X) = (U_1^*(X), U_2^*(X), ..., U_p^*(X))$ is an optimal partition for $X \in P(N)^p$ if it is a partition of the set of user vertices that verifies:

$$v_i \in U_k^*(X) \Rightarrow w'_i d(x_k, v_i) = \min_{x \in X} w'_i d(x, v_i).$$

Given an optimal partition of the user vertices set U, the p-center and the

p-median objective functions can be computed by:

$$f_c(U;X) = \max_{k=1,\dots,p} f_c(U_k^*(X);x_k).$$

$$f_m(U;X) = \sum_{k=1}^{p} f_m(U_k^*(X);x_k).$$

Given a candidate solution X and the corresponding optimal partition $U^*(X)$, associated with each x_k there is a radius $r_k(X)$ which represents the farthest distance from x_k to $U_k^*(X)$:

$$r_k(X) = \max_{v_i \in U_k^*(X)} w_i' d(x_k, v_i).$$

Then

$$f_c(U;X) = \max_{k=1,\dots,p} r_k(X) = r^*(X),$$

and

$$\begin{aligned} f_\lambda(U;X) &= \lambda \max_{k=1,\dots,p} r_k(X) + (1-\lambda) f_m(U;X) = \\ &= \lambda r^*(X) + (1-\lambda) f_m(U;X). \end{aligned}$$

Assume that $x_k \notin EP$ for some $k \in \{1, ..., p\}$. This means also that $x_k \notin V \cup LC \cup PP \cup BP$. We are going to show that the set X can be modified without increasing the function $f_\lambda(U;X)$ until $x_k \in D$ for every k. We consider two cases: $r_k(X) < r^*(X)$ and $r_k(X) = r^*(X)$.

Case 1. Let $r_k(X) < r^*(X)$.

Since $x_k \notin V$, it must be an interior point of an edge $[i,j]$. We intend show how to move this point x_k on its edge without increasing the function $f_\lambda(U;X)$ until a vertex is reached or $r_k(X)$ equals $r^*(X)$. In order to do so, we will analyse the slope of $f_\lambda(U;X)$ in terms of the distance from x_k to the end vertex i of its edge $[i,j]$.

For any interior point x of an edge $[i,j]$, the set of user vertices which are optimally reached from x through i is denoted by $U^i(x)$; i.e.,

$$U^i(x) = \{v_r \in U : w_r' d(x, v_r) = w_r'(l(x,i) + d(i, v_r)) \leq w_r'(l(x,j) + d(j, v_r))\}.$$

Analogously,

$$U^j(x) = \{v_r \in U : w'_r d(x, v_r) = w'_r(l(x, j)+d(j, v_r)) \leq w'_r(l(x, i)+d(i, v_r))\}.$$

Given the set X of facility points, let $U_k^=(X)$ be the set of user vertices that can be assigned to x_k and also to other x_m, for some $m \neq k$; i.e.,

$$U_k^=(X) = \{v_i \in U : \quad w'_i d(x_k, v_i) = \min_{x \in X} w'_i d(x, v_i) \text{ and } w'_i d(x_k, v_i) = w'_i d(x_m, v_i), \text{ for some } m \neq k\}.$$

The user vertices assigned to x_k that are not assignable to another facility point are those of $U_k^<(X) = U_k(X) - U_k^=(X)$; i.e.,

$$U_k^<(X) = \{v_i \in U : \quad w'_i d(x_k, v_i) = \min_{x \in X} w'_i d(x, v_i) \text{ and } w'_i d(x_k, v_i) < w'_i d(x_m, v_i), \text{ for every } m \neq k\}.$$

The vertices of $U_k^=(X)$ are assignable to x_k and to another facility point in X, but if x_k is moved a small amount ξ towards j or towards i some of these vertices can not remain assigned to x_k and some of them must be assigned only to x_k because the tie is destroyed. Let the new point $x_k(\xi)$ denote x_k when it is moved an amount ξ towards j; i.e., if $x_k = p([i, j], t)$ then $x_k(\xi) = p([i, j], t + \xi)$. Then, those user vertices of $U_k^=(X) \cap U^j(x_k)$ are assigned only to $x_k(\xi)$ and those of $U_k^=(X) - U^j(x_k)$ are no assigned to $x_k(\xi)$.

Let $X(\xi) = (x_1, x_2, ..., x_k(\xi), ..., x_p)$. Then the m-th component of the new optimal partition for $X(\xi)$ is given by:
$U_m^*(X(\xi)) = U_m^*(X) - [U_k^=(X) \cap U^j(x_k)]$, for every $m \neq k$, and
$U_k^*(X(\xi)) = U_k^*(X) - [U_k^=(X) - U^j(x_k)]$.

The slope of $f_m(U; X(\xi))$, as a function of ξ, depends of the user vertices assignable to x_k (those of $U_k(X(\xi)) = U_k^<(X) \cup U_k^=(X)$) that are optimally reached or not through j. The value of this slope is:

$$s_m^j(\xi) = w(U_k^<(X) - U^j(x_k)) - w([U_k^<(X) \cup U_k^=(X)] \cap U^j(x_k)).$$

Where

$$w(W) = \sum_{v_i \in W} w_i, \forall W \subset U.$$

When the movement is towards the vertex i the new facility point is $x_k(-\xi) = p(e, t-\xi)$ and, analogously, $X(-\xi) = (x_1, x_2, ..., x_k(-\xi), ..., x_p)$. The slope of $f_m(U; X(-\xi))$, as a function of ξ, is:

$$s_m^i(\xi) = w(U_k^<(X) - U^i(x_k)) - w([U_k^<(X) \cup U_k^=(X)] \cap U^i(x_k)).$$

One of these slopes is not positive since:

$$s_m^i(\xi) + s_m^j(\xi) = -2w([U_k^<(X) \cup U_k^=(X)] \cap U^i(x_k) \cap U^j(x_k)) \leq 0.$$

Let us assume (the other case is similar) that $s_m^j(\xi) \leq 0$.

The slope $s_m^j(\xi)$ could change from non-positive to positive only if one of the following cases holds: *a*) the set $U_k^<(X) - U^j(x_k)$ gets a vertex, or *b*) the set $[U_k^<(X) \cup U_k^=(X)] \cap U^j(x_k)$ loses a vertex. Let us analyse them.

a) $U_k^<(X) - U^j(x_k)$ gets a vertex. The only possibilities for this are:

   a1) A vertex leaves $U^j(x_k(\xi))$, but this is not possible because we are moving $x_k(\xi)$ towards j.

   a2) A vertex that is not in $U^j(x_k(\xi))$ comes into $U_k^<(X)$, but this is not possible either because, for this vertex, the distance to $x_k(\xi)$ increases while the distance to x_m, for $m \neq k$, does not change.

b) $[U_k^<(X) \cup U_k^=(X)] \cap U^j(x_k)$ loses a vertex. This is impossible because the distance from $x_k(\xi)$ to the vertices of $U^j(x_k)$ decreases as $x_k(\xi)$ gets closer to j, then they can not leave $U_k^<(X) \cup U_k^=(X)$.

While $r_k(X(\xi)) \leq r^*(X(\xi)) = r^*(X)$ the slope of f_c is zero. Then, the slope of f_λ is: $s_\lambda^j(\xi) = (1-\lambda) \cdot s_m^j(\xi)$, if the movement is towards j, and $s_\lambda^i(\xi) = (1-\lambda) \cdot s_m^i(\xi)$, if the movement is towards i. The sum or these two values is:

$$s_\lambda^j(\xi) + s_\lambda^i(\xi) = \underbrace{(1-\lambda)}_{\geq 0} \underbrace{(s_m^i(\xi) + s_m^j(\xi))}_{\leq 0} \leq 0.$$

So, in any case one of the slopes is not positive. This means that we can always move x_k in the direction which has no positive slope where f_λ does not increase, until it reaches a vertex or the corresponding radius r_k equals r^*. In this last case we will be in case 2.

Case 2. Let $r_k(X) = r^*(X)$.

We study two subcases: $2a$ and $2b$. In case $2a)$ $r_m(X) < r_k(X)$, for all $m \neq k$; i.e., $r^*(X) = \max_{i\in\{1,..,p\}} r_i(X)$ is equal only to $r_k(X)$. And in case $2b)$ $r_m(X) = r_k(X)$, for some $m \neq k$; i.e., $r^*(X) = \max_{i\in\{1,..,p\}} r_i(X)$ is equal to several of $r_i(X)$, $i \in \{1, .., p\}$.

Case $2a)$ $r_m(X) < r_k(X) = r^*(X)$, for all $m \neq k$.

As in case 1, let us analyse the slope of function f_λ when moving $x_k \in [i, j]$ an amount ξ towards j or towards i. Let $x_k(\xi)$ denote x_k when it is moved an amount ξ towards j and $x_k(-\xi)$ denote x_k when it is moved an amount ξ towards i.

Let $U^* = \{v_i \in U_k^*(X) : w_i' d(x_k, v_i) = r^*(X)\}$. Since $x_k \notin EP$, it is not a local center then only the following two cases are possible:

a) $U^* \subset U^j(x_k)$ and $U^* \cap U^i(x_k) = \emptyset$ then
$r^*(X(\xi)) = r_k(X(\xi)) = r_k(X) - w_i'\xi$.

b) $U^* \subset U^i(x_k)$ and $U^* \cap U^j(x_k) = \emptyset$ then
$r^*(X(\xi)) = r_k(X(\xi)) = r_k(X) + w_i'\xi$.

Therefore the slope of $f_c(U; X(\xi))$ is $+w_i'$ or it is $-w_i'$, (and the slope of $f_c(U; X(-\xi))$ is $-w_i'$ or w_i') respectively. Thus the slope of f_λ is exppresed in one of the following ways:

a) $s_\lambda(\xi) = s_\lambda^j(\xi) = -\lambda w_i' + (1-\lambda) s_m^j(\xi)$ and
$s_\lambda(-\xi) = s_\lambda^i(\xi) = +\lambda w_i' + (1-\lambda) s_m^i(\xi)$.

b) $s_\lambda(\xi) = s_\lambda^j(\xi) = +\lambda w_i' + (1-\lambda) s_m^j(\xi)$ and
$s_\lambda(-\xi) = s_\lambda^i(\xi) = -\lambda w_i' + (1-\lambda) s_m^i(\xi)$.

In both cases, as can be deduced from the analysis of case 1, the sum of the slope when moving x_k towards j plus the slope when moving it towards i is:

$$s_\lambda^j(\xi) + s_\lambda^i(\xi) = \underbrace{(1-\lambda)}_{\geq 0} \underbrace{(s_m^i(\xi) + s_m^j(\xi))}_{\leq 0} \leq 0.$$

Therefore, moving x_k in one of the directions (towards i or towards j) the function f_λ does not increase until one of the following cases occurs:

i) $x_k(\xi)$ becomes a vertex (i or j), a local center or a pendant point with respect to two users vertices from U^*, where the slope of $f_c(U_k(X(\xi)); x_k(\xi))$ could be $+w'_i$ in both directions, or

ii) $r_k(X(\xi))$ becomes equal to some r_m, with $m \neq k$. In this last case we will be in case $2b$).

Case $2b$) $r_m(X) = r_k(X) = r^*(X)$, for some $m \neq k$.

Let K be the set of indices where $r^*(X)$ is reached; i.e., $k \in K$ if and only if $r_k(X) = r^*(X)$. If for some $k^* \in K$, $x_{k^*} \in LC(r^*) \cup PP(r^*) \cup BP(r^*) \cup V$, then $r^* \in R$ and $x_k \in EP(r^*)$, $\forall k \in K$. Otherwise $x_k \notin LC(r^*) \cup PP(r^*) \cup BP(r^*) \cup V$, $\forall k \in K$. Let us consider this case.

For every $k \in K$, the facility point x_k is interior to an edge and $x_k \notin LC(r^*) \cup PP(r^*) \cup BP(r^*)$. By the analysis of case $2a$, we can denote the edge containing x_k by $[i_k, j_k]$ and by w'_k the weight of the vertex in $U_k(x)$ farthest from x_k, and in the case that it was not unique, it will be the maximum of such weights, in such way that the slope of $f_c(U_k(X); x_k)$ is $+w'_k$ when moving x_k towards j_k and it is $-w'_k$ when moving x_k towards i_k (the interchange of the names i_k and j_k could be necessary). In this last case w'_k is the weight of the vertex in $U_k(x)$ farthest from x_k, and in the case that it was not unique, it will be the minimum of such weights. We will move at the same time all the points x_k, $k \in K$, an amount ξ/w'_k on their edges in the direction in which simultaneously every r_k increases or decreases. Notice that the weight w'_m of x_m with $m \neq k$ will be w'_k, because otherwise x_m will be a pendant point.

The slope of the objective function f_λ is:

a) If every x_k, for $k \in K$, is moved $\frac{\xi}{w'_k}$ towards j_k:

$$s_\lambda(\xi) = +\lambda \; + \; (1-\lambda) \sum_{k \in K} \frac{1}{w'_k} s_k^{j_k}(\xi).$$

b) If every x_k, for $k \in K$, is moved $\frac{\xi}{w'_k}$ towards i_k:

$$s_\lambda(-\xi) = -\lambda \; + \; (1-\lambda) \sum_{k \in K} \frac{1}{w'_k} s_k^{i_k}(\xi).$$

One of these two slopes must be non positive, because the sum of them is:

$$(1-\lambda) \quad \sum_{k\in K} \frac{1}{w'_k} \quad \underbrace{\left[s_k^{i_k}(\xi) + s_k^{j_k}(\xi)\right]}_{\leq 0} \quad \leq 0.$$

Therefore, moving at the same time all the x_k, $k \in K$, in the corresponding directions (towards the i_k's or towards the j_k's) the function f_λ does not increase until happens:

i) that some x_k reach the some of the extremes of its edge (i_k or j_k) or

ii) that some x_k become a pendant point or local center with respect to two vertices of U^* with range $r_k(X(\xi)) = r^*(X(\xi))$ where the slope of $f_c(U_k(X(\xi)); x_k(\xi))$ is $+w'_k$ in both directions, or

iii) that $r_k(X(\xi))$ becomes equal to some r_m, with $m \notin K$.

In this last case a new index comes into K and the process is iterated. This completes the proof. □

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Algorithms.

The finite dominating set EP provides a rudimentary procedure for solving the problem: an exhaustive search in the set of all combinations of p points of EP. The complexity of this algorithm depends on the size of EP. Let n be the size of the vertex set V and m be the number of edges.

Since $|EP^p| = O(n^{3p}m^{2p})$, an exact algorithm for the generalized p-facility λ-Centdian problem on a network N based in an exhaustive search in EP^p has complexity $O(n^{3p+1}m^{2p})$, because the objective function is computed with complexity $O(n)$.

The following result is

A consequence of the proof of the theorem and allows us to reduce the complexity of the search in the set of candidates. Let the set of local centers, pendant points, bottleneck points, vertices, and extreme points with range r be denoted by

$$D(r) = V \cup LC \cup PP \cup BP \cup EP(r).$$

Then there is an optimal solution X^* for the problem such that, if $r^* = f_c(U; X^*)$ is the maximum radius of the solution, then $X^* \subseteq D(r^*)$.

We can reduce the search for an optimal solution to the set of combinations of p points of $D(r)$, for each $r \in R$. Since $|D(r)| = O(n^2 m)$ for every r, these $|R|$ searches of the combinations of p points imply evaluating $O(n^{2p+2} m^{p+1})$ candidate solutions. The objective function is computed in $O(n)$ time, so the complexity of the exact algorithm is $O(n^{2p+3} m^{p+1})$.

The complexity of the search can be reduced even more by observing that one of the points in X^* has to be a local center, a pendant point or a vertex that determines the value of r^*, and the other $p-1$ points of X^* have to be vertices or extreme points with range r^*.

Proposition 0.7. *Let X^* be an optimal solution for the generalized p-λ-Centdian problem, and $r^* = f_c(U; X^*)$ the maximum radius. Then, exist a point $x^* \in X^*$ such that $x^* \in LC(r^*) \cup BP(r^*)$, $x^* \in PP(r^*)$, or that $x^* \in VV(r)$, moreover, the other $p-1$ points of X^* are vertices or extreme points with range r^*.*

Proof: Its follows from the proof of the above theorem. □

Proposition 0.8. *Let $r^* = r^*(U; X^*)$, $r_c^* = r^*(X_c)$ and $r_m^* = r^*(X_m)$ be the radii of the generalized p-λ-Centdian, p-center and p-median respectively. Then $r_c^* \leq r^* \leq r_m^*$.*

Proof: Its follows from the proof of the above theorem. □

The proposed algorithm is as follows:

Algorithm p-CD-G.

- **Step 1.** Obtain the list L of pairs (x, r) consisting in local centers, pendant points bottleneck points x and its respective radii r and vertices in $v \in VV(r)$.
- **Step 2.** Compute the p-center X_c and the p-median X_m. Let $r_c^* = r^*(X_c)$ and $r_m^* = r^*(X_m)$ be the corresponding maximum radii.
- **Step 3.** For every $(x, r) \in L$ with $r_c^* \leq r \leq r_m^*$ do the following.
  - i) Obtain the set $EP(r)$ of extreme points with range r, and
  - ii) For every selection Y of $p-1$ points of $V \cup EP(r)$ compute $f_\lambda(U; \{x\} \cup Y)$.
- **Step 4.** Keep the best of the sets of p points evaluated in step 3.

Proposition 0.9. *The complexity of the algorithm p-CD-G is* $O(n^{p+2}m^p)$.

Proof: Let DR be the set of pairs (*point*,*range*) given by

$$DR = \{(x, r) : x \in LC(r) \cup PP(r) \cup BP(r) \cup VV(r)\}$$

For each $(x, r) \in DR$ we only need to search for solutions consisting of x and $p-1$ points in $V \cup EP(r)$. Then $|DR| = O(n^2m)$ since $|LC| = O(n^2m)$, $|PP(r)| = O(n^2m)$, $|BP(r)| = O(nm)$ and $|V| = O(n)$. Moreover $|EP(r)| = O(nm)$ for every $r \geq 0$. All this results in $O(n^2m)$ searches in $(V \cup EP(r))^{p-1}$ to find candidate sets of p points. Evaluating each candidate set takes a time $O(n)$. Thus the exact algorithm has complexity $O(n^2m \cdot (nm)^{p-1} \cdot n) = O(n^{p+2}m^p)$. □

If the underlying network is a tree there is a polynomial algorithm due to Tamir, Pérez-Brito and Moreno Pérez (1998).

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Conclusions

The first step in order to solve a continuous location problems is usually to find a finite dominant set (see Hooker et al. (1991)). For the p-median problem, the set of vertices is a finite dominating set. For the p-center problem also local center have to be considered. For the unweighted the p-Centdian problem also extreme points have to be included. In this paper we show that for the generalized p-Centdian problem we need to consider also the pendant points. So while the problem is being more complicated we need to include new points in the dominating set but we got a finite one.

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