Full ordering Voronoi sets for Bi-facility bicriterium (Max,Sum) location on networks *

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Abstract

The Bi-facility Max-Sum Location problem on networks consists in finding a pair of points on the network minimizing the maximum and the sum of the distances of this pairs to a finite set of user points. A solution is Pareto-optimal if no other solution is better for the two objectives. It is a Bayes-optimal solution if it is optimal for some linear combination of the objectives. The analysis of the distance function allows us to identify a kind of Voronoi sets where the order of the user points from the farthest one to the nearest one does not change. We show how to use these sets to geometrically construct the set of pairs of values of the objective functions for all the solutions. This set consists of the union of triangles where the bidimensional objective is a linear function. The dominance between the vertices and sides of these triangles can be used to identify all the Pareto-optimal and Bayes-optimal solutions of the problem. **Keywords:**Location, Networks, Voronoi Sets, Centdian.

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1 Introduction

Facility location deals with problems of locating one or several facilities in order to optimize some criteria with regard to the users. Multi-facility problems on networks consist of selecting several points of a network in order to minimize a function which is distance dependent with respect to given user points of the network. The median and the center problems are two well known location problems with numerous applications. The first is suitable for locating a facility providing a routine service, by means of minimizing the average distances of users to it. The second is appropriate for emergency services where the objective is to have the farthest users as near as possible to the center.

In many real world problems the objective is a mixture of these different, possibly adverse objectives; i.e. the minimization of the travelling time to the farthest potential user and also being as close as possible to the heavy demand zones. The problem is therefore a bicriteria problem consisting of selecting the locations that minimize both objective functions. An efficient or Pareto-optimal solution is a selection of location points such that there is no other possible solution with better value for both objectives; this is a basic approach in multicriteria optimization. The goal may also be mathematically expressed by minimizing a new objective function that is a linear combination of the objective functions of the center and median problems. A weighted or Bayes-optimal is an optimal to some linear combination of the objectives. This multi-objective approach for locating a facility on a network was introduced by Halpern [3] and [4], who coined the term centdian for the point which minimizes the convex combination of the center and median objective functions. A finite dominating set for the multifacility centdian on network has been recently found in [7].

Next section gives the basic definitions and notations. In section 3 we formulate the centdian problem on a network. In section 4, we define the breaking points we need to obtain a kind of Voronoi sets where the distances to the user points have the same order. These sets are used to get the image of the possible solutions in the space of the objective values as union of polygons. The final section indicates how to use these results to geometrically construct the image space of the possible solutions and the set of Pareto-optimal and the set of Bayes-optimal solutions.

2 Definitions and Notation

We give the detailed definitions and notations for a rigorous mathematical formulation of the problem. A graph G consists of a set of points arranged in a finite set V of single points named *vertices* or **nodes** and a finite set E of continuous and linear set of points named *arcs* or **edges**. Every edge joins two nodes and there is no other node in the edge. A network N consists in a pair (G, l) where G is a graph and l is a function that gives the positive bounded length $l(e)$ of every edge e in E. The length of an edge is a measure of its size The edge joining nodes i and j is denoted by $e = [i, j]$ and its length is denoted by $l(e) = l(i, j) = l(j, i)$. Every point x on edge $[i, j]$ is determined by a value t, $0 \leq t \leq l(i, j)$, which represents the length of the portion of the edge between i and x. This point is denoted by $x = p([i, j], t) = p([j, i], l(i, j) - t)$. For every two points x and y on a same edge $e = [i, j]$, the portion of the edge between x and y is a *subedge* of the network denoted by $[x, y]$. If we consider the points $x = p([i, j], t)$ and $y = p([i, j], s)$ then the length of the subedge $[x, y]$ is $l(x, y) = |t - s|$.

For each edge $e = [i, j]$, the only nodes in e are i and j, which are the *extremes* of the edge, and the remainder of its points are *interior* to the edge. The *insertion* of an interior point x of the edge $[i, j]$ in the network N consists in transforming it into an equivalent network $N + x$ where x is a new node and the edge $[i, j]$ is replaced by the subedges $[i, x]$ and $[x, j]$ as new edges of the network. If $x = p([i, j], t)$ then the lengths of the new edges $[i, x]$ and $[x, j]$ are t and $l(i, j) - t$, respectively. When this point is inserted in N, the points $p([i, j], s)$ that were interior of $[i, j]$ in N are now denoted in $N + x$ by $p([i, x], s)$, when $s \leq t$, and by $p([x, j], s - t)$, when $s \geq t$. If x is a node of N then $N + x = N$. The *extraction* of v consists of transforming it into an equivalent network $N - v$ where v is eliminated as a node and the edges $[i, v]$ and $[v, j]$ are replaced by a new edge $[i, j]$. The length of this new edge is $l(i, j) = l(i, v) + l(v, j)$. When this node is extracted from N, a point x that was interior to $[i, j]$ in N is now denoted in $N - v$ by $p([i, j], l(i, v) - l(x, v))$, if x belongs to $[v, i]$, and by $p([i, j], l(i, v) + l(x, v))$, if x belongs to $[v, j]$, while v is now the interior point $p([i, j], l(i, v))$

A *path* between two nodes of a graph is a finite sequence of edges joining them. Every path P between two nodes i and j of N is given by a sequence of nodes $i_0, i_1, ..., i_r$ such that $i = i_0$, $j = i_r$ and $[i_{k-1}, i_k] \in E$, for $k = 1, ..., r$.

The *length* $l(p)$ of the path P is equal to the sum of the lengths of all these edges.

$$l(P) = \sum_{k=1}^{r} l(i_{k-1}, i_k).$$

A path between two points x and y of the network is a path between the nodes x and y of the network $N + x + y = N + \{x, y\}$. The distance $d(x, y)$ between any two points x and y on N is the minimum length of a path between them. This distance is a metric on the set of points of the network N.

The network N, as a set of points with a distance function, is a topological metric space that can be described by several equivalent models each one consisting of a finite set of nodes and a finite set of edges joining them with the corresponding lengths. These different models are obtained by inserting and extracting a finite set of points. The distance function given by the shortest path length is a metric independent of the model chosen.

3 The p-centdian problem

In facility location on networks, the length of each edge represents the cost of going once through it to satisfy the demand of one user. The distance between points represents the cost of the cheapest way of going from one point to the other to supply one user. The users are assumed to be located at a finite set of points and each facility can be open at any point of the network.

For every integer p, the *p-facility location problem* consists of choosing p of the potential locations to establish a facility at them and allocating every user to one of the facilities such that an objective function that depends on the distances between the users and the facilities to which they are allocated is optimized [5]. Two of the most classical p-facility location problems are the *p-center* problem and the *p-median* problem [1]. In both of them every user is allocated to the nearest facility point. The p-median problem consists of determining the locations that minimizes the sum of distances between all the users and their facility point. The objective of the p-center problem is to minimize the maximum distance from any facility point to any of the users assigned to it.

Let U be the finite set of points of N representing the users. Let $X = (x_1, ..., x_p)$ be a vector of p points of N representing a *solution* consisting of

p the facility points. The distance from X to an user point $u \in U$ is given by:

$$d(X,u) = \min_{x \in X} d(x,u) = \min\{d(x_1,u), d(x_2,u), ..., d(x_p,u)\}.$$

Let all solutions consisting of p points of N be denoted by N^p. Given the network N and the set U of user points the p-median problem is to find the solution $X^* \in N^p$ that minimizes the objective function:

$$F_m(X) = \sum_{u \in U} d(X,u).$$

The p-center problem is to find the solution $X^* \in N^p$ that minimizes the objective function:

$$F_c(X) = \max_{u \in U} d(X,u).$$

¿From a multicriteria point of view, both objective functions are to be minimized, leading to the bicriteria or biobjective *p-centdian problem* Let the set of objective values be: $F(N^p) = \{(F_m(X), F_c(X)) : X \in N^p\}$. A pair of objective values $f = (f_m, f_c)$ dominates the pair $f' = (f'_m, f'_c)$ if $f_m \leq f'_m$ and $f_c \leq f'_c$, with at least one strict inequality. A pair of objective values $f = (f_m, f_c) \in F(N^p)$ is a Pareto-optimal pair of values if it is not dominated by another pair $f' = (f'_m, f'_c) \in F(N^p)$. Let P denote the set of Pareto-optimal pairs of values. The solution X is *Pareto-optimal* if $F(X) \in P$. The solution X is *Bayes-optimal* if there are non negative w_m and w_c and not both equal zero such that X minimizes the objective function $w_m \cdot F_m + w_c \cdot F_c$ in N^p. Let Q denote the pairs of objective values of the Bayes-optimal solutions; i.e. the set of Bayes-optimal pairs of values. Thus $f = (f_m, f_c) \in F(N^p)$ is a Bayes-optimal pair if, for any $w_m, w_c \geq 0$ and not both equal zero, there exists no other pair $f' = (f'_m, f'_c) \in F(N^p)$ such that $w_m \cdot f'_m + w_c \cdot f'_c < w_m \cdot f_m + w_c \cdot f_c$. Our main goal is to geometrically construct the sets P and Q. Note that Q is a subset of P.

In the case $p = 2$, we have the problem:

$$\min_{x_1, x_2 \in N} \{F_c(x_1, x_2), F_m(x_1, x_2)\}.$$

Let $F : N^2 \to I\!R^2$ be the function from the set of pairs of points of the network to the plane, defined as:

$$F(x_1, x_2) = (F_c(x_1, x_2), F_m(x_1, x_2)).$$

The objective space is the image $F(N^2) = \{F(x_1, x_2) : x_1, x_2 \in N\}$. The set P consists of those pairs that belong to the lower-left boundary of $F(N^2)$ and Q consists of those pairs that also belong to the lower boundary of the convex hull of $F(N^2)$ or P. It will be shown in the next section that $F(N^2)$ is not convex and its boundary is piecewise linear.

4 The Breaking Points.

Since E is the set of edges of the network, $F(N^2)$ can be obtained by:

$$F(N^2) = \bigcup_{e_1, e_2 \in E} F(e_1, e_2).$$

where $F(e_1, e_2) = \{F(x_1, x_2) : x_1 \in e_1, x_2 \in e_2\}$. The main objective of this section is to identify a finite set of points of the network to be inserted in order to get easy expressions for $F(e_1, e_2)$.

Definition 1 *A point $x \in N$ is a* **breaking point** *with respect to user points $u, v \in U$ if x is an interior point of an edge $[i, j]$ of $N + U$ such that:*

$$d(u, x) = d(u, i) + l(i, x) = l(x, j) + d(j, v) = d(x, v)$$

When $u = v$ the breaking point is usually named *bottleneck point* and when $u \neq v$ it is a *local center*. These points are useful for solving the p-center problem on networks (see [6]). Note also that every user point is a breaking point by considering $x = i = j = u = v$.

Let B be the set of breaking points with respect to user points of U. Let $M = N + B$ be the network obtained by inserting in N all the breaking points. Then $W = V \cup B$ is the set of nodes of the new network M and let $E[M]$ denote the set of edges of this network. Our first lemma shows that the edges of this network are the *full ordering Voronoi sets* of the network in the sense that, for every edge $e \in E[W]$, all its points have the same order of the user points from the farthest to the nearest one. Hakimi and Labbé [2] considered other Voronoi sets in location theory.

Lemma 2 *Let e be an edge of $E[W]$. Then, for every $u, v \in U$, we have:*

$$\forall x, y \in e : d(x, u) \leq d(x, v) \iff d(y, u) \leq d(y, v).$$

Proof. Thank to lemma 1

$$\frac{d(j,u)-d(i,u)}{l(i,j)} = \begin{cases} +1 & \text{if } d(j,u) > d(i,u) \\ -1 & \text{if } d(j,u) < d(i,u) \end{cases}$$

Let $e = [i,j]$ be an edge of $E[W]$. For every $u \neq v \in U$, if $d(i,u) < d(i,v)$ and $d(j,u) > d(j,v)$ then there is a point x interior to $[i,j]$ such that $d(x,u) = d(x,v)$. Then x is a local center in the interior of $[i,j]$ with respect to u and v. However this is not possible if all the local centers are nodes of $N + W$. $\square$.

Now we show that, since all bottleneck points are inserted as nodes of $N + W$, the distances to the user points are linear function with slope $+1$ or -1 on every edge of $E[W]$; i.e., between two adjacent points of W.

Lemma 3 *For every edge $e = [i,j] \in E[W]$ and every $u \in U$ we have:*

$$d(p([i,j],t),u) = d(i,u) + t \cdot \frac{d(j,u)-d(i,u)}{l(i,j)}, \forall t \in [0, l(i,j)].$$

Proof. Let $[i,j] \in E[W]$ and $u \in U$. Then $\forall x \in [i,j]$ we have $d(x,u) = \min\{l(x,i)+d(i,u), l(x,j)+d(j,u)\}$. If $l(x,i)+d(i,u) = l(x,j)+d(j,u)$ then x is a bottleneck point with respect to u. Since there is not an interior point of $[i,j]$ that is a bottleneck point then $d(x,u) = l(x,i) + d(i,u)$, $\forall x \in [i,j]$, or $d(x,u) = l(x,j) + d(j,u)$, $\forall x \in [i,j]$. If $x = p([i,j],t)$ then $l(x,i) = t$ and $l(x,j) = l(i,j) - t$. Thus $d(p([i,j],t),u) = d(i,u) + t$, if $d(i,u) < d(j,u)$, and $d(p([i,j],t),u) = d(i,u) - t$, if $d(i,u) > d(j,u)$. Therefore $d(p([i,j],t),u)$ is the linear function of t with slope $+1$ or -1 between $d(i,u)$ for $t = 0$ and $d(j,u)$ for $t = l(i,j)$ shown in the lemma. $\square$.

The edges of $E[W]$ are linearity regions for the distance from a point of the network to a user point. We look now for linearity regions for the distance from a pair of points of the network to a user point.

Lemma 4 *Let e_1 and e_2 be two edges of $E[W]$. For every user point $u \in U$, the distance to u is a piecewise linear function with at most two regions in the set of pairs (x_1, x_2) with $x_1 \in e_1, x_2 \in e_2$.*

Proof. Denote by $x_1 = p([i_1,j_1],t_1)$ with $0 \leq t_1 \leq l(i_1,j_1)$ and $x_2 = p([i_2,j_2],t_2)$ with $0 \leq t_2 \leq l(i_2,j_2)$. Then $d(\{x_1,x_2\},u)$ is giving by

$$\min\{d(i_1,u) + t_1 \cdot \frac{d(j_1,u)-d(i_1,u)}{l(i_1,j_1)}, d(i_2,u) + t_2 \cdot \frac{d(j_2,u)-d(i_2,u)}{l(i_2,j_2)}\}.$$

Therefore the function $g(t_1, t_2) = d(\{x_1, x_2\}, u) = d(\{p([i_1, j_1], t_1), p([i_2, j_2], t_2)\}, u)$ defined for (t_1, t_2) in the rectangle $[0, l(i_1, j_1)] \times [0, l(i_2, j_2)]$ is a linear function in each one of the two regions of this rectangle defined by the straight line with slope $+1$ or -1 given by:

$$d(i_1, u) + t_1 \cdot \frac{d(j_1, u) - d(i_1, u)}{l(i_1, j_1)} = d(i_2, u) + t_2 \cdot \frac{d(j_2, u) - d(i_2, u)}{l(i_2, j_2)}.$$

□.

In the generic case, this line divides the rectangle in two quadrilaterals. Each quadrilateral could degenerate in a triangle while the other one becomes a pentagon. Even one of these triangles can become the empty set; then the other region is the whole rectangle. If the edges have the same length both quadrilaterals can be triangles and then the rectangle is a square which is divided in the two regions of linearity by one of its diagonals. Namely, let the edges e_1 and e_2 be given by $e_1 = [i_1, j_1]$ and $e_2 = [i_2, j_2]$ where the names of the nodes are chosen in such way that $l_1 = l(i_1, j_1) \geq l_2 = l(i_2, j_2)$ and such that $d_1 = d(i_1, u) < d(j_1, u) = d_1 + l_1$ and $d_2 = d(i_2, u) < d(j_2, u) = d_2 + l_2$. Then (see figure 1) we have:

- If $d(i_1, u) \leq d(i_2, u) \leq d(j_2, u) \leq d(j_1, u)$ then the straight line divides the rectangle $[0, l_1] \times [0, l_2]$ in the two quadrilaterals; one with vertices $(0, 0)$, $(0, l_2)$, $(d_2 - d_1, 0)$ and $(d_2 + l_2 - d_1, l_2)$ and the other one with vertices $(l_1, 0)$, (l_1, l_2), $(d_2 - d_1, 0)$ and $(d_2 - d_1 + l_2, l_2)$.

- If $d(i_1, u) \leq d(i_2, u) \leq d(j_1, u) \leq d(j_2, u)$ then the two regions are the pentagon with vertices $(0, 0)$, $(0, l_2)$, (l_1, l_2) $(d_2 - d_1, 0)$ and $(l_1, d_1 + l_1 - d_2)$ and the triangle with vertices $(l_1, 0)$, $(0, d_2 - d_1)$ and $(l_1, d_1 + l_1 - d_2)$.

- If $d(i_1, u) \leq d(j_1, u) \leq d(i_2, u) \leq d(j_2, u)$ the first region is the whole rectangle and the second region is the empty set.

Any other possibility of inequalities between these distances is equivalent to one of them by interchanging the role of the nodes i_1, i_2, j_1 and j_2.

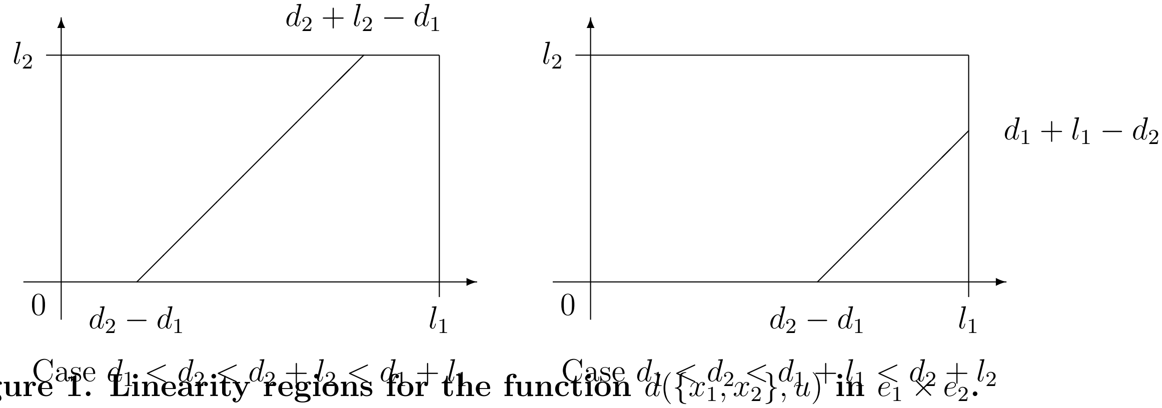

Figure 1. Linearity regions for the function $d(\{x_1, x_2\}, u)$ in $e_1 \times e_2$.

Lemma 5 *Let $[i_1, j_1], [i_2, j_2] \in E[W]$ such that $d(i_1, u) < d(j_1, u)$ and $d(i_2, u) < d(j_2, u)$. Let $x_1 = p([i_1, j_1], t_1)$ and $x_2 = p([i_2, j_2], t_2)$. Then*

$$d(\{x_1, x_2\}, u) = \begin{cases} d(i_1, u) + t_1 & \textit{for} \quad t_1 + d(i_1, u) \le t_2 + d(i_2, u) \\ d(i_2, u) + t_2 & \textit{for} \quad t_1 + d(i_1, u) \ge t_2 + d(i_2, u) \end{cases}$$

Proof. Let $x_1 = p([i_1, j_1], t_1)$ with $0 \le t_1 \le l(i_1, j_1)$ and $x_2 = p([i_2, j_2], t_2)$. with $0 \le t_2 \le l(i_2, j_2)$. Since $[i_1, j_1]$ and $[i_2, j_2]$ are edges in $E[W]$, if $d(i_1, u) < d(j_1, u)$ then $d(x_1, u) = d(i_1, u) + t_1$ and if $d(i_2, u) < d(j_2, u)$ then $d(x_2, u) = d(i_2, u) + t_2$. Therefore

$$d(\{x_1, x_2\}, u) = \min\{d(x_1, u), d(x_2, u)\} = \min\{d(i_1, u) + t_1, d(i_2, u) + t_2\}.$$

Thus the expression of the lemma follows. $\square$.

Now, the sets $F(e_1, e_2)$ are easy to compute, for every two edges $e_1, e_2 \in E[W]$. The set of user points U is partitioned in the sets U_1, U_2 and U_0 defined by:

$$U_1 = \{u \in U : d(x_1, u) \le d(x_2, u), \forall x_1 \in e_1, x_2 \in e_2\}$$

$$U_2 = \{u \in U : d(x_1, u) \ge d(x_2, u), \forall x_1 \in e_1, x_2 \in e_2\}$$

$$U_0 = U \setminus (U_1 \cup U_2)$$

First we consider the case in which $U_0 = \emptyset$ and later when $U_0 \neq \emptyset$.

Let $e_1 = [i_1, j_1]$ and $e_2 = [i_2, j_2]$ be two edges of $E[W]$. Next lemma shows that if $U_0 = \emptyset$ then $F(e_1, e_2)$ is the convex hull of the points $F(i_1, i_2), F(i_1, j_2), F(j_1, i_2)$

and $F(j_1, j_2)$. This is usually a quadrilateral with vertices $F(i_1, i_2), F(i_1, j_2), F(j_1, i_2)$ and $F(j_1, j_2)$, however it could also be a triangle or even a segment. Let $H(S)$ denote the convex hull of the set S.

Lemma 6 *Let $e_1, e_2 \in E[W]$. If, for every two interior points x_1 of e_1 and x_2 of e_2,*

$$d(x_1, u) \neq d(x_2, u), \forall u \in U$$

then

$$F(e_1, e_2) = H(\{F(i_1, i_2), F(i_1, j_2), F(j_1, i_2), F(j_1, j_2)\}).$$

Proof. Let $l_1 = l(e_1) = l(i_1, j_1)$ and $l_2 = l(e_2) = l(i_2, j_2)$. Then

$$F(e_1, e_2) = \{F(x_1, x_2) : x_1 \in e_1, x_2 \in e_2\} =$$

$$\{F(p([i_1, j_1], t_1), p([i_2, j_2], t_2)) : t_1 \in [0, l_1], t_2 \in [0, l_2]\}.$$

Note that if $d(x_1, u) \neq d(x_2, u)$, for every user node $u \in U$ and for every two interior points x_1 of e_1 and x_2 of e_2, then $U_0 = \emptyset$. Let $ij_1 = p([i_1, j_1], l_1/2)$ and $ij_2 = p([i_2, j_2], l_2/2)$. Then

$$U_1 = \{u \in U : d(ij_1, u) < d(ij_2, u)\}$$

and

$$U_2 = \{u \in U : d(ij_1, u) > d(ij_2, u)\}.$$

Then $U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2 = U$. Moreover $\forall x_1 \in e_1, x_2 \in e_2$ we have: 1) if $u \in U_1$ then $d(x_1, u) \leq d(x_2, u)$, and if $u \in U_2$ then $d(x_1, u) \geq d(x_2, u)$.

Let the center and median functions with respect to a subset of user points $U' \subseteq U$ be defined, for every network point x, by:

$$F_m^{U'}(x) = \sum_{u \in U'} d(x, u) and F_c^{U'}(x) = \max_{u \in U'} d(x, u)$$

. Let us analyze first the objective of the center problem F_c. We know that $F_c(x_1, x_2) = \max\{F_c^{U_1}(x_1), F_c^{U_2}(x_2)\}$ and $F_c^{U_1}(x_1) \neq F_c^{U_2}(x_2)$. Without loss of optimality we may assume that $F_c^{U_1}(x_1) > F_c^{U_2}(x_2)$. The opposite case is however never mentioned, but should be, be it only by a symmetry argument. Then by lemma 1 some $u_c \in U_1$ exists such that $F_c^{U_1}(x) = d(u_c, x)$ for all $x \in e_1$, and the formula then follows from lemma 2.

$$F_c(x_1, x_2) = F_c^{U_1}(x_1) = F_c^{U_1}(i_1) + \frac{t_1}{l_1}(F_c^{U_1}(j_1) - F_c^{U_1}(i_1)).$$

since due to all assumptions $F_c(i_1, i_2) = F_c(i_1) = F_c^{U_1}(i_1)$, therefore:

$$F_c(x_1, x_2) = F_c(i_1, i_2) + \frac{t_1}{l_1}(F_c(j_1, i_2) - F_c(i_1, i_2)).$$

Let us now analyze the objective of the median problem F_m. Moreover, $\forall x_1 \in e_1, x_2 \in e_2$ is $F_m(x_1, x_2) = F_m^{U_1}(x_1) + F_m^{U_2}(x_2)$. Also

$$F_m^{U_1}(x_1) = F_m^{U_1}(i_1) + \frac{t_1}{l_1}(F_m^{U_1}(j_1) - F_m^{U_1}(i_1))$$

and

$$F_m^{U_2}(x_2) = F_m^{U_2}(i_2) + \frac{t_2}{l_2}(F_m^{U_2}(j_2) - F_m^{U_2}(i_2)).$$

Therefore $F_m(x_1, x_2)$ is

$$F_m^{U_1}(i_1) + F_m^{U_2}(i_2) + \frac{t_1}{l_1}(F_m^{U_1}(j_1) - F_m^{U_1}(i_1)) + \frac{t_2}{l_2}(F_m^{U_2}(j_2) - F_m^{U_2}(i_2)).$$

Moreover

$$F_m^{U_1}(j_1) - F_m^{U_1}(i_1) = F_m(j_1, i_2) - F_m(i_1, i_2) = F_m(j_1, j_2) - F_m(i_1, j_2)$$

and

$$F_m^{U_2}(j_2) - F_m^{U_2}(i_2) = F_m(i_1, j_2) - F_m(i_1, i_2) = F_m(j_1, j_2) - F_m(j_1, i_2).$$

Since due to all assumptions $F_m(i_1, i_2) = F_m(i_1) = F_m^{U_1}(i_1)$ thus:

$$F_m(x_1, x_2) = F_m(i_1, i_2) + \frac{t_1}{l_1}(F_m(j_1, i_2) - F_m(i_1, i_2)) + \frac{t_2}{l_2}(F_m(j_1, j_2) - F_m(j_1, i_2))$$

and

$$F_m(x_1, x_2) = F_m(i_1, i_2) + \frac{t_1}{l_1}(F_m(j_1, j_2) - F_m(i_1, j_2)) + \frac{t_2}{l_2}(F_m(i_1, j_2) - F_m(i_1, i_2)).$$

In order to see both objective at the same time, note that, since $\forall x_2 \in [i_2, j_2]$ is $F_c(i_1, x_2) = F_c(i_1, i_2) = F_c(i_1, j_2)$ and $F_c(j_1, x_2) = F_c(j_1, i_2) = F_c(j_1, j_2)$ then also:

$$F_c(x_1, x_2) = F_c(i_1, i_2) + \frac{t_1}{l_1}(F_c(j_1, i_2) - F_c(i_1, i_2)) + \frac{t_2}{l_2}(F_c(j_1, j_2) - F_c(j_1, i_2)).$$

and

$$F_c(x_1,x_2) = F_c(i_1,i_2) + \frac{t_1}{l_1}(F_c(j_1,j_2) - F_c(i_1,j_2)) + \frac{t_2}{l_2}(F_c(i_1,j_2) - F_c(i_1,i_2)).$$

Thus finally:

$$F(x_1,x_2) = F(i_1,i_2) + \frac{t_1}{l_1}(F(j_1,i_2) - F(i_1,i_2)) + \frac{t_2}{l_2}(F(j_1,j_2) - F(j_1,i_2))$$

and also

$$F(x_1,x_2) = F(i_1,i_2) + \frac{t_1}{l_1}(F(j_1,j_2) - F(i_1,j_2)) + \frac{t_2}{l_2}(F(i_1,j_2) - F(i_1,i_2)).$$

Therefore $F(e_1,e_2) = \{F(x_1,x_2) : x_1 \in [i_1,j_1], x_2 \in [i_2,j_2]\}$ is the quadrilateral with vertices $F(i_1,i_2)$, $F(i_1,j_2)$, $F(j_1,i_2)$ and $F(j_1,j_2)$ that correspond to (t_1,t_2) being $(0,0)$, $(0,l_2)$, $(l_1,0)$ and (l_1,l_2). $\square$.

The proof of lemma 5 may be summarized by expliciting the fact that on the set $e_1 \times e_2$ both the functions F_c and F_m are linear. By the way, this also explains why the result is a (possibly degenerated) quadrilateral.

We now analyze the case in which $U_0 \neq \emptyset$. Consider the network $M = N + W$. For any pair of user points $u,v \in U$ and any node $k \in W$ let us insert any interior point x with $d(x,u) = d(k,v)$ as a new node. Let C be the set of these points; i.e.,

$$C = \{x \in N : d(x,u) = d(k,v), k \in W, u,v \in U\}.$$

Consider now the network $M + C$ obtained by inserting in M all the points in C as new nodes. Let $E[M + C]$ denote the set of edges of this network.

Note that there is no interior point x of an edge in $E[M+C]$ such that the distance from x to a user point is equal to the distance from a node $w \in W$ to another user point. Therefore for any two edges $e_1,e_2 \in E[M + C]$ either all distance functions to user points are linear on $e_1 \times e_2$, or $l(e_1) = l(e_2)$ and each distance function is linear on some half of the square $e_1 \times e_2$, obtained by division by one of its diagonals. Thus, if for some user point $u \in U$, $d(x_1,u) = d(x_2,u)$ for two interior points $x_1 \in e_1$ and $x_2 \in e_2$ then the lengths of the edges coincides; $l(e_1) = l(e_2)$.

If $e_1,e_2 \in E[C]$ have the same length $l(e_1) = l(e_2) = l$ then

$$F(e_1,e_2) = \{F(p([i_1,j_1],t_1), p([i_2,j_2],t_2)) : t_1,t_2 \in [0,l]\}.$$

Then we consider the square $[0,l]^2$ divided by its two diagonals in the four sets:

$$S_1 = \{(t_1,t_2) \in [0,l]^2 : t_1 \le t_2 \le l - t_1\}$$
$$S_2 = \{(t_1,t_2) \in [0,l]^2 : l - t_1 \le t_2 \le t_1\}$$
$$S_3 = \{(t_1,t_2) \in [0,l]^2 : t_2 \le t_1 \le l - t_2\}$$
$$S_4 = \{(t_1,t_2) \in [0,l]^2 : l - t_2 \le t_1 \le t_2\}$$

These are the four triangles with vertices:

$$\{(0,0),(0,l),(l/2,l/2)\}$$
$$\{(l,l),(l,0),(l/2,l/2)\}$$
$$\{(0,0),(l,0),(l/2,l/2)\}$$
$$\{(l,l),(0,l),(l/2,l/2)\}.$$

By the next lemma we will show that the image of each of these triangles is also a possibly degenerated triangle determined by the images of the corresponding vertices.

Since by interchanging the role of the nodes i_1, j_1, i_2 and j_2 these are equivalent triangles we only need to consider the set $T_1 = \{F(p([i_1,j_1],t_1),p([i_2,j_2],t_2)) : (t_1,t_2) \in S_1\}$ to prove that this is the triangle

$$T_1 = H(\{F(i_1,i_2),F(i_1,j_2),F(ij_1,ij_2)\})$$

that has the vertices corresponding to the pairs of values: $\{(0,0),(0,l),(l/2,l/2)\}$.

Lemma 7 *Let $e_1,e_2 \in E[C]$ with $l(e_1) = l(e_2) = l$. Then*

$$H_1 = \{F(x_1,x_2) = F(p([i_1,j_1],t_1),p([i_2,j_2],t_2) : 0 \le t_1 \le t_2 \le l - t_1\}$$

is the triangle with vertices $F(i_1,i_2)$, $F(i_1,j_2)$ and $F(ij_1,ij_2)$ where $ij_1 = p([i_1,j_1],l/2)$ and $ij_2 = p([i_2,j_2],l/2)$.

Proof. Let $l(e_1) = l(i_1,j_1) = l(e_2) = l(i_2,j_2) = l$. Let $x_1 = p([i_1,j_1],t_1)$ and $x_2 = p([i_2,j_2],t_2)$ with $0 \le t_1 \le t_2 \le l - t_1$. Note that

$$d(\{x_1,x_2\},u) = \min\{t_1+d(i_1,u), l-t_1+d(j_1,u), t_2+d(i_2,u), l-t_2+d(j_2,u)\}.$$

Let the partition of U in the four sets given by:

$$U_1^- = \{u \in U : d(j_1, u) + l = d(i_1, u) < d(i_2, u)\}$$

$$U_2^- = \{u \in U : d(j_2, u) + l = d(i_2, u) \leq d(i_1, u)\}$$

$$U_1^+ = \{u \in U : d(j_1, u) + l > d(i_1, u) \leq d(i_2, u)\}$$

$$U_2^+ = \{u \in U : d(j_2, u) + l > d(i_2, u) < d(i_1, u)\}.$$

Then:

- If $u \in U_1^-$ then $d(\{x_1, x_2\}, u) = d(x_1, u) = d(i_1, u) - t_1$.
- If $u \in U_2^-$ then $d(\{x_1, x_2\}, u) = d(x_2, u) = d(i_2, u) - t_2$.
- If $u \in U_1^+$ then $d(\{x_1, x_2\}, u) = d(x_1, u) = d(i_1, u) + t_1$.
- If $u \in U_2^+$ then $d(\{x_1, x_2\}, u) = d(x_2, u) = d(i_2, u) + t_2$.

Let us analyze first the objective of the center problem F_c. Let

$$U^* = \{u \in U : d(\{i_1, i_2\}, u) = F_c(\{i_1, i_2\})\}.$$

Then

- If $U^* \subseteq U_2^-$ then $F_c(\{x_1, x_2\}) = F_c(\{i_1, i_2\}) - t_2$.
- If $U^* \subseteq U_1^- \cup U_2^-$ and $U^* \cap U_1^- \neq \emptyset$ then $F_c(\{x_1, x_2\}) = F_c(\{i_1, i_2\}) - t_1$.
- If $U^* \subseteq U_1^- \cup U_2^- \cup U_1^+$ and $U^* \cap U_1^+ \neq \emptyset$ then $F_c(\{x_1, x_2\}) = F_c(\{i_1, i_2\}) + t_1$.
- If $U^* \cap U_2^+ \neq \emptyset$ then $F_c(\{x_1, x_2\}) = F_c(\{i_1, i_2\}) + t_2$.

Therefore, in all cases, the function $G_c(t_1, t_2) = F_c(\{p([i_1, j_1], t_1), p([i_2, j_2], t_2))\}$ is a linear function in S_1.

Let us now analyze the objective of the median problem F_m.

$$F_m(\{x_1, x_2\}) = \sum_{u \in U} d(\{x_1, x_2\}, u) =$$

$$= \sum_{u \in U_1^-} d(\{x_1, x_2\}, u) + \sum_{u \in U_2^-} d(\{x_1, x_2\}, u) + \sum_{u \in U_1^+} d(\{x_1, x_2\}, u) + \sum_{u \in U_2^+} d(\{x_1, x_2\}, u) =$$

$$= \sum_{u \in U_1^-} (d(i_1, u) - t_1) + \sum_{u \in U_2^-} (d(i_2, u) - t_2) + \sum_{u \in U_1^+} (d(i_1, u) + t_1) + \sum_{u \in U_2^+} (d(i_2, u) + t_2) =$$

$$= \sum_{u \in U} d(\{i_1, i_2\}, u) + t_1 \cdot (|U_1^+| - |U_1^-|) + t_2 \cdot (|U_2^+| - |U_2^-|)$$

Therefore, the function $G_m(t_1, t_2) = F_m(\{p([i_1, j_1], t_1), p([i_2, j_2], t_2))\}$ is also a linear function in S_1.

Thus, if $G(t_1,t_2) = (G_c(t_1,t_2), G_m(t_1,t_2))$ then the set

$$H_1 = G(S_1) = \{G(t_1,t_2) : (t_1,t_2) \in S_1\} =$$

$$= \{F(x_1,x_2) : x_1 = p([i_1,j_1],t_1), x_2 = p([i_2,j_2],t_2, \text{ with } 0 \leq t_1 \leq t_2 \leq l - t_1\}$$

is the triangle with vertices $F(i_1,i_2)$, $F(i_1,j_2)$ and $F(ij_1,ij_2)$ that correspond to (t_1,t_2) being $(0,0)$, $(0,l)$ and $(l/2,l/2)$; i.e.,

$$H_1 = H(G(0,0), G(0,l), G(l/2,l/2)).$$

$\square$.

5 Conclusions

Note that in any case, for every two edges $e_1, e_2 \in E[C]$, the set $F(e_1,e_2)$ is the union of the four triangles:

$$T_1 = H(\{F(i_1,i_2), F(i_1,j_2), F(ij_1,ij_2)\})$$

$$T_2 = H(\{F(i_1,i_2), F(j_1,i_2), F(ij_1,ij_2)\})$$

$$T_3 = H(\{F(i_1,j_2), F(j_1,j_2), F(ij_1,ij_2)\})$$

$$T_4 = H(\{F(j_1,i_2), F(j_1,j_2), F(ij_1,ij_2)\}).$$

So we obtain the set $F(N^2)$ as a union of triangles of which every side represents a set of solutions where one of its points is in C and the other one is moving along an edge of $E[C]$ or both points in the solution are moving at the same time by the same amount on two respective edges with the same length.

The set of Pareto-optimal pairs of objective values consists of the sides of all these triangles of which both vertices are nondominated by other vertices (of all the triangles). The set of Bayes-optimal pairs of objective values is the lower boundary of the convex hull of the set of the vertices of all these triangles.

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