Structure-Preserving Nonholonomic Integrators

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Numerical Integration of Nonholonomic Systems
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Outline

1. Lagrangian Mechanics with Symmetry
   - Lagrangian Mechanics
   - Systems with Symmetry
   - Nonholonomic Dynamics
2. Discrete Nonholonomic Systems On Lie Groups
   - Discrete Mechanics
   - Discrete Systems on Lie Groups

Lagrangian Mechanics

- $L : TQ \to \mathbb{R}$ is the Lagrangian (kinetic minus potential energy), where Q is the configuration space
- $q(t)$ is a trajectory if and only if $q(t)$ extremizes the action $\int_a^b L(q(t), \dot{q}(t))\, dt$
- If nonholonomic constraints are present, the variation are taken *before* imposing the constraints

Momentum Map

- $\Phi : G \times Q \to Q$ is a free and proper left action of a Lie group G on Q
- The Lagrangian $L : TQ \to \mathbb{R}$ is invariant with respect to $\Phi_* : G \times TQ \to TQ$
- $\eta_Q(q) \in TQ$ is the infinitesimal generator associated with the Lie algebra element $\eta \in \mathfrak{g}$
- The momentum map $J : TQ \to \mathfrak{g}^*$ is defined by $\langle J, \eta \rangle = \left\langle \frac{\partial L}{\partial \dot{q}}, \eta_Q \right\rangle$
- $J = \text{const}$ throughout the motion

Nonholonomic Momentum

- $\dot{q} \in \mathscr{D}_q \subset T_q Q$
- The constraint distribution $\mathscr{D} = \cup_q \mathscr{D}_q$ is left-invariant
- $\mathscr{S}_q := \mathscr{D}_q \cap T_q \operatorname{Orb}(q)$
- $\mathfrak{g}^q := \{\eta \in \mathfrak{g} \mid \eta_Q(q) \in \mathscr{S}_q\}$
- $J^{\mathrm{nhc}} : T_q Q \to (\mathfrak{g}^q)^*$ is defined by $\left\langle J^{\mathrm{nhc}}, \eta \right\rangle = \left\langle \frac{\partial L}{\partial \dot{q}}, \eta_Q \right\rangle$ for $\eta \in \mathfrak{g}^q$
- J^{nhc} *generically is not a constant of motion*
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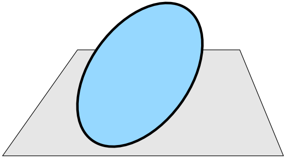
- 

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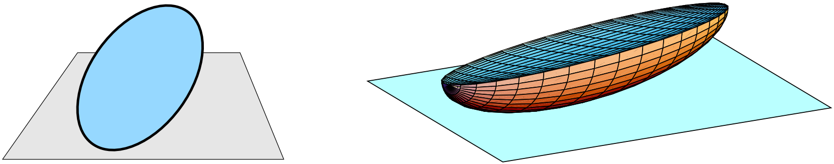
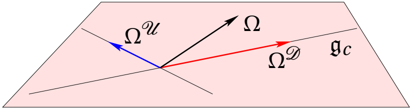


Nonholonomic Dynamics

- $(r, g) \in Q/G \times G, \quad \xi := g^{-1}\dot{g}$
- $\Omega := \xi + \mathscr{A}\dot{r}$ is the *body angular velocity*, the constraints read $\Omega \in \mathfrak{g}_c$



- $$l(r, \dot{r}, \Omega) = \frac{1}{2}\big(\langle M(r)\dot{r}, \dot{r}\rangle + \langle I_{\mathscr{D}}(r)\Omega^{\mathscr{D}}, \Omega^{\mathscr{D}}\rangle + \langle I_{\mathscr{U}}(r)\Omega^{\mathscr{U}}, \Omega^{\mathscr{U}}\rangle\big) + \langle \Lambda_{\mathscr{U}}(r)\dot{r}, \Omega^{\mathscr{U}}\rangle - U(r)$$
- The nonholonomic momentum relative to the body frame is $p := \dfrac{\partial l}{\partial \Omega^{\mathscr{D}}} \in \mathfrak{g}_c^*$
- $$\frac{d}{dt}\frac{\partial l}{\partial \dot{r}} - \frac{\partial l}{\partial r} = -\left\langle p, \frac{\partial I_{\mathscr{D}}^{-1}}{\partial r} p \right\rangle - \langle p + \mathbf{i}_{\dot{r}}\lambda, \mathbf{i}_{\dot{r}}\mathscr{B} + \langle \mathbf{i}_{\dot{r}}\gamma, \mathscr{A}\rangle - \langle \gamma, \mathbf{i}_{\dot{r}}\mathscr{A}\rangle + \langle \mathscr{E}, I_{\mathscr{D}}^{-1} p\rangle\rangle,$$
  $$\dot{p} = \Big[\mathrm{ad}^*_{I_{\mathscr{D}}^{-1}p}\, p + \big(\mathrm{ad}^*_{I_{\mathscr{D}}^{-1}p}\, \mathbf{i}_{\dot{r}}\lambda + \langle p, \mathbf{i}_{\dot{r}}\mathscr{E}\rangle\big) + \langle \mathbf{i}_{\dot{r}}\lambda, \mathbf{i}_{\dot{r}}\mathscr{E}\rangle\Big]_{\mathscr{D}}$$

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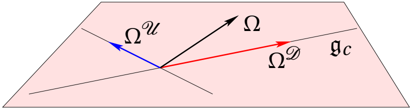

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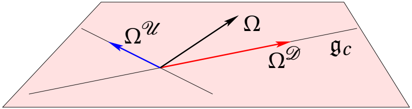

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Momentum Conservation in the Body Frame

- For simplicity, assume $\langle \mathbf{i}_{\dot r}\lambda, \mathbf{i}_{\dot r}\mathscr{E}\rangle\big|_{\mathscr{D}} = 0$
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Theorem

$\langle p, \eta(r)\rangle$ is a conservation law if and only if

- *the distribution defined by $\mathrm{ad}^*_{I_{\mathscr{D}}^{-1}p}\,\mathbf{i}_{\dot r}\,\lambda + \langle p, \mathbf{i}_{\dot r}\mathscr{E}\rangle$ is flat*
- *$\Big\langle \big[\mathrm{ad}^*_{I_{\mathscr{D}}^{-1}p}\, p\big]_{\mathscr{D}}, \eta\Big\rangle = 0$ for an arbitrary value of the nonholonomic momentum p*

- These integrals often cannot be written explicitly
- They are useful in qualitative analysis, such as justifying quasiperiodic behavior of the falling penny and of the Routh problem
- They are useful in stability analysis (such as the energy-momentum method)
- For systems on Lie groups, $\eta(r)$, if exists, is a constant vector from $\mathfrak{g}$

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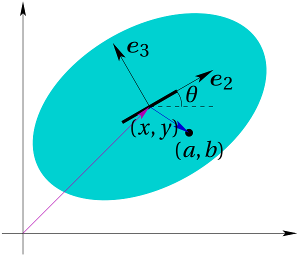
The Chaplygin Sleigh

- $G = SE(2)$
- e_1 is directed towards the reader
- $\mathfrak{g}_c = \mathrm{span}\{e_1, e_2\}$
- $\dot{p}_1 = -ma\Omega^1\Omega^2, \quad \dot{p}_2 = ma(\Omega^1)^2$
- $$\Omega^1 = \frac{(M+m)p_1 + mbp_2}{(M+m)(J+ma^2)+Mmb^2}$$
  $$\Omega^2 = \frac{mbp_1 + \left(J+ma^2+mb^2\right)p_2}{(M+m)(J+ma^2)+Mmb^2}$$
- If $a = 0$, the body momentum is conserved: $p_1 = \text{const}$, $p_2 = \text{const}$



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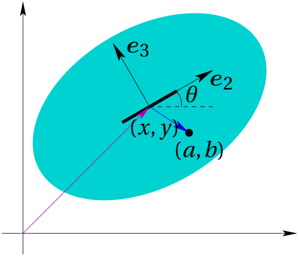
- The contact point travels either along a circle (generically), or along a straight line



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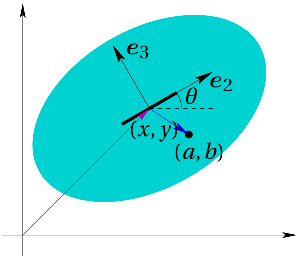

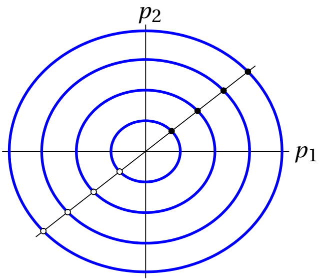

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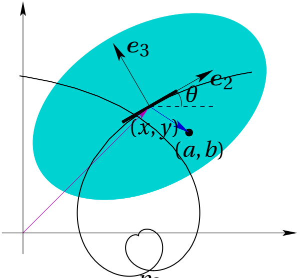
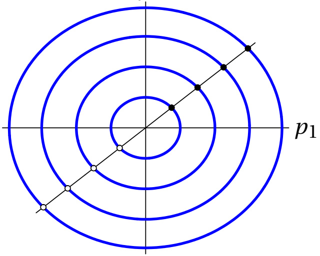
- The contact point moves along a curve with a cusp





Discrete Mechanics

- The discrete Lagrangian $L_d : Q \times Q \to \mathbb{R}$ approximates the action integral along an exact solution of the Euler–Lagrange equations joining q_k and q_{k+1}
- In the discrete setting, the action integral of Lagrangian mechanics is replaced by an action sum $S_d = \sum_{k=0}^{N-1} L_d(q_k, q_{k+1})$
- Taking the extremum over $q_1, \ldots, q_{N-1}$ gives the discrete Euler–Lagrange equations $D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = 0$
- This discretization is structure-preserving: The discrete dynamics preserves momentum, symplectic form, etc.

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- In the nonholonomic setting one needs (Cortés and Martínez)
  - A constraint distribution $\mathcal{D} = \{\dot{q} \in TQ \mid \langle A(q)^j, \dot{q}\rangle = 0, j = 1, \ldots, m\}$
  - A *discrete constraint space* $\mathcal{D}_d \subset Q \times Q$
- The discrete dynamics becomes

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = \sum_{j=1}^{m} \lambda_{j,k} A^j(q_k), \quad (q_k, q_{k+1}) \in \mathcal{D}_d$$

- Other discretization models were studied, see e.g. M. de León, D. Martín de Diego, and A. Santamaría-Merino
- Discrete mechanics on Lie grupoids, see J. Marrero, D. Martín de Diego, and E. Martínez

Discrete Systems on Lie Groups

- The configuration space is a Lie group G; $L_d : G \times G \to \mathbb{R}$ and $\mathscr{D}_d \subset G \times G$ are invariant with respect to the left diagonal action of G on $G \times G$; $\mathscr{D}$ is left-invariant with respect to the induced action of G on TG
- Define the incremental displacements $W_k \in G$ by the formula $W_k = g_k^{-1} g_{k+1}$
- $L_d(g_k, g_{k+1}) = l_d(W_k)$, where $l_d : G \to \mathbb{R}$ is the reduced discrete Lagrangian
- $(g_k, g_{k+1}) \in \mathscr{D}_d$ if and only if $W_k \in \mathscr{S}_d \subset G$, where $\mathscr{S}_d$ is the reduced discrete constraint space
- $\mathscr{D} = \cup_g L_{g*} \mathfrak{g}_c$, where $\mathfrak{g}_c = \{\xi \in \mathfrak{g} \mid \langle a^1, \xi \rangle = \cdots = \langle a^m, \xi \rangle = 0\}$ is a subspace of $\mathfrak{g}$
- The discrete body momentum is $p_k := R_{W_k}^* l_d'(W_k)$
- Recall that the nonholonomic momentum in the continuous-time case belongs to a subspace $\mathfrak{g}_c^*$ of the Lie algebra $\mathfrak{g}$
- In the discrete setting $p_k \in \mathscr{U}_d = \mathscr{L}(\mathscr{S}_d)$, where $\mathscr{L}$ is the discrete Legendre transform; $\mathscr{U}_d$ is not always a subspace of $\mathfrak{g}^*$

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- The configuration space is a Lie group G; $L_d : G \times G \to \mathbb{R}$ and $\mathscr{D}_d \subset G \times G$ are invariant with respect to the left diagonal action of G on $G \times G$; $\mathscr{D}$ is left-invariant with respect to the induced action of G on TG
- Define the incremental displacements $W_k \in G$ by the formula $W_k = g_k^{-1} g_{k+1}$
- $L_d(g_k, g_{k+1}) = l_d(W_k)$, where $l_d : G \to \mathbb{R}$ is the reduced discrete Lagrangian
- $(g_k, g_{k+1}) \in \mathscr{D}_d$ if and only if $W_k \in \mathscr{S}_d \subset G$, where $\mathscr{S}_d$ is the reduced discrete constraint space
- $\mathscr{D} = \cup_g L_{g*} \mathfrak{g}_c$, where $\mathfrak{g}_c = \{\xi \in \mathfrak{g} \mid \langle a^1, \xi \rangle = \cdots = \langle a^m, \xi \rangle = 0\}$ is a subspace of $\mathfrak{g}$
- The discrete body momentum is $p_k := R^*_{W_k} l'_d(W_k)$
- Recall that the nonholonomic momentum in the continuous-time case belongs to a subspace $\mathfrak{g}^*_c$ of the Lie algebra $\mathfrak{g}$
- In the discrete setting $p_k \in \mathscr{U}_d = \mathscr{L}(\mathscr{S}_d)$, where $\mathscr{L}$ is the discrete Legendre transform; $\mathscr{U}_d$ is not always a subspace of $\mathfrak{g}^*$

Discrete Systems on Lie Groups

Theorem (Fedorov and Zenkov [2005], McLachlan and Perlmutter [2006])

The discrete Euler–Lagrange equations

$$D_1L_d(g_k, g_{k+1}) + D_2L_d(g_{k-1}, g_k) = \sum_{j=1}^{m} \lambda_{j,k} A^j(g_k), \quad (g_k, g_{k+1}) \in \mathscr{D}_d$$

are equivalent to the discrete Euler–Poincaré–Suslov equations

$$p_{k+1} - \mathrm{Ad}^*_{W_k} p_k = \sum_{j=1}^{s} \lambda_{j,k+1} a^j, \quad W_k \in \mathscr{S}_d$$

coupled with the discrete reconstruction equation

$$g_{k+1} = g_k W_k$$

How should one define $\mathscr{S}_d$?

1. $\mathscr{S}_d := \{W_k \in G \mid \log W_k \in \mathfrak{g}_c\}$ $((\log W_k)/h \to \Omega$ as $h \to 0$)
2. $\mathscr{S}_d := \mathscr{L}^{-1}(\mathfrak{g}_c^*)$ or $p_k \in \mathfrak{g}_c^*$ ($p \in \mathfrak{g}_c^*$ in the continuous-time case)

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The Discrete Chaplygin Sleigh

- $G = SE(2)$
- $W = \begin{pmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ 0 & 0 & 1 \end{pmatrix}$
- $l_d(W_k) = hl\big((W_k - e)/h\big), \quad \mathcal{S}_d = \left\{ W \in SE(2) \,\middle|\, \frac{W_{23}}{W_{13}} = \frac{1 - W_{11}}{W_{21}} \right\}$

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- If $a = 0, \ p_{k+1} = p_k$

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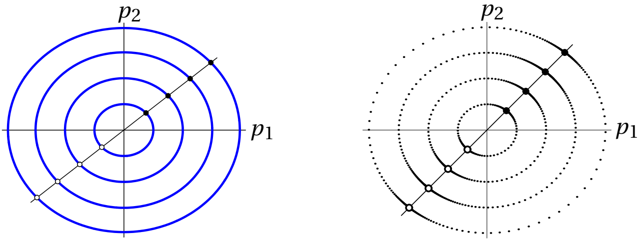
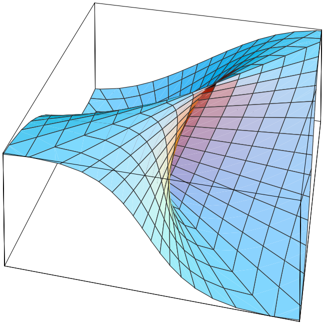
- If $a \neq 0$, the evolution of the discrete nonholonomic momentum is in excellent agreement with that of the continuous-time momentum, and the constrained energy is preserved



The Discrete Chaplygin Sleigh

- $p_k \in \mathscr{U}_d = \mathscr{L}(\mathscr{S}_d)$
- For the Chaplygin sleigh, $\mathscr{U}_d$ is diffeomorphic to the Möbius strip embedded in $\mathfrak{g}^* = se^*(2)$



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- The discrete Euler–Poincaré–Suslov equations become
  $p_{1,k+1} = p_{1,k} + p_{2,k} W_{23,k}$, $p_{2,k+1} = p_{2,k} W_{11,k}$
- The body momentum is preserved if and only if $W_k = \begin{pmatrix} 1 & 0 & C \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (motion of a sleigh along a straight line). So, this choice of the discrete constraint space does not result in a structure-preserving algorithm

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Summary

- Nonholonomic system have unusual momentum conservation laws
- Nonholonomic integrators are structure-preserving provided that the discrete constraint space was properly selected
- Work in progress
  - Discrete systems with internal degrees of freedom
  - Discrete systems with right-invariant constraints