Construction of Variational Integrators on Lie Groupoids

Numerical integration of nonholonomic systems

Eduardo Martínez
University of Zaragoza
`emf@unizar.es`

Madrid, December 21, 2006

Abstract

Interesting mechanical systems defined on the tangent bundle (of course) but also on Lie algebras, principal fibre bundles, integrable subbundles (distributions), semidirect products, etc.

One can develop general framework to describe all those systems as particular cases by using the geometry of Lie algebroids.

When discretizing one obtains a general framework to study the discrete analogs of such systems.

One can also introduce nonholonomic constraints (Juan Carlos's talk)

Lie Algebroids

A Lie algebroid structure on the vector bundle $\tau\colon E \to M$ is given by

- a Lie algebra structure $(\mathrm{Sec}(E), [\ ,\ ])$ on the set of sections of E, and
- a morphism of vector bundles $\rho\colon E \to TM$ over the identity, such that
  - $\rho([\sigma, \eta]) = [\rho(\sigma), \rho(\eta)]$
  - $[\sigma, f\eta] = f[\sigma, \eta] + (\rho(\sigma)f)\,\eta$,

  where $\rho(\sigma)(m) = \rho(\sigma(m))$.

The first condition is actually a consequence of the second and the Jacobi identity.

Examples

- **Tangent bundle.**

  $E = TM$,

  $\rho = \mathrm{id}$,

  $[\,,\,] =$ bracket of vector fields.

- **Integrable subbundle.**

  $E \subset TM$, integrable distribution

  $\rho = i$, canonical inclusion

  $[\,,\,] =$ restriction of the bracket to vector fields in E.

- **Lie algebra.**

  $E = \mathfrak{g} \to M = \{e\}$, Lie algebra (fiber bundle over a point)
  $\rho = 0$, trivial map (since $TM = \{0_e\}$)
  $[\,,\,] =$ the bracket in the Lie algebra.

- **Atiyah algebroid.**

  Let $\pi\colon Q \to M$ a principal G-bundle.
  $E = TQ/G \to M$, (Sections are equivariant vector fields)
  $\rho([v]) = T\pi(v)$ induced projection map
  $[\,,\,] =$ bracket of equivariant vectorfields (is equivariant).

Transformation Lie algebroid.

Let $\Phi\colon \mathfrak{g} \to \mathfrak{X}(M)$ be an action of a Lie algebra $\mathfrak{g}$ on M.

$E = M \times \mathfrak{g} \to M$,

$\rho(m, \xi) = \Phi(\xi)(m)$ value of the fundamental vectorfield

$[\,,\,] =$ induced by the bracket on $\mathfrak{g}$.

Mechanics on Lie algebroids

(Weinstein 1996, Martínez 2001)

(De León, Marrero, Cortés, etc.)

Lie algebroid $E \to M$.

$L \in C^\infty(E)$ or $H \in C^\infty(E^*)$

- ☐ $E = TM \to M$ Standard classical Mechanics
- ☐ $E = \mathcal{D} \subset TM \to M$ (integrable) System with holonomic constraints
- ☐ $E = TQ/G \to M = Q/G$ System with symmetry
- ☐ $E = \mathfrak{g} \to \{e\}$ System on a Lie algebra
- ☐ $E = M \times \mathfrak{g} \to M$ System on a semidirect product (ej. heavy top)

Structure functions

A local coordinate system (x^i) in the base manifold M and a local basis of sections (e_α) of E, determine a local coordinate system (x^i, y^α) on E.

The anchor and the bracket are locally determined by the local functions $\rho^i_\alpha(x)$ and $C^\alpha_{\beta\gamma}(x)$ on M given by

$$\rho(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i}$$

$$[e_\alpha, e_\beta] = C^\gamma_{\alpha\beta}\ e_\gamma.$$

The function ρ^i_α and $C^\alpha_{\beta\gamma}$ satisfy some relations due to the compatibility condition and the Jacobi identity which are called the structure equations:

$$\rho^j_\alpha \frac{\partial \rho^i_\beta}{\partial x^j} - \rho^j_\beta \frac{\partial \rho^i_\alpha}{\partial x^j} = \rho^i_\gamma C^\gamma_{\alpha\beta}$$

$$\sum_{\mathrm{cyclic}(\alpha,\beta,\gamma)} \left[\rho^i_\alpha \frac{\partial C^\nu_{\beta\gamma}}{\partial x^i} + C^\mu_{\beta\gamma} C^\nu_{\alpha\mu} \right] = 0.$$

Exterior differential

On 0-forms

$$df(\sigma) = \rho(\sigma) f$$

On p-forms ($p > 0$)

$$\begin{aligned} d\omega(\sigma_1, \ldots, \sigma_{p+1}) = \\ &= \sum_{i=1}^{p+1} (-1)^{i+1} \rho(\sigma_i) \omega(\sigma_1, \ldots, \widehat{\sigma_i}, \ldots, \sigma_{p+1}) \\ &- \sum_{i<j} (-1)^{i+j} \omega([\sigma_i, \sigma_j], \sigma_1, \ldots, \widehat{\sigma_i}, \ldots, \widehat{\sigma_j}, \ldots, \sigma_{p+1}). \end{aligned}$$

Exterior differential-local

Locally determined by

$$dx^i = \rho^i_\alpha e^\alpha$$

and

$$de^\alpha = -\frac{1}{2} C^\alpha_{\beta\gamma} e^\beta \wedge e^\gamma .$$

The structure equations are

$$d^2 x^i = 0 \quad \text{and} \quad d^2 e^\alpha = 0.$$

Lagrange's equations

Given a function $L \in C^\infty(E)$, we define a dynamical system on E by means of a system of differential equations, which in local coordinates reads

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) + \frac{\partial L}{\partial y^\gamma} C^\gamma_{\alpha\beta} y^\beta = \rho^i_\alpha \frac{\partial L}{\partial x^i}\\ &\dot{x}^i = \rho^i_\alpha y^\alpha.\end{aligned}$$

The equation $\dot{x}^i = \rho^i_\alpha y^\alpha$ is the local expression of the admissibility condition: A curve $a \colon \mathbb{R} \to E$ is said to be admissible if

$$\rho \circ a = \frac{d}{dt}(\tau \circ a).$$

In other words, if $a\,dt \colon T\mathbb{R} \to E$ is an admissible map.

Prolongation

Given a Lie algebroid $\tau\colon E \to M$ we can construct the E-tangent to E (the prolongation of E). It is the vector bundle $\tau_1\colon \mathcal{T}^E E \to E$ where the fiber over $a \in E$ is

$$\mathcal{T}_a^E E = \{\, (b, v) \in E_m \times T_a E \mid T\tau(v) = \rho(b) \,\}$$

where $m = \tau(a)$.

Redundant notation: (a, b, v) for the element $(b, v) \in \mathcal{T}_a^E E$.

The bundle $\mathcal{T}^E E$ can be endowed with a structure of Lie algebroid. The anchor $\rho^1 \colon \mathcal{T}^E E \to TE$ is just the projection onto the third factor $\rho^1(a, b, v) = v$.

The structure of Lie algebroid in $\mathcal{T}^E E$ can be defined in terms of the brackets of vertical and complete lifts

$$[\eta^C, \sigma^C] = [\sigma, \eta]^C, \qquad [\eta^C, \sigma^V] = [\sigma, \eta]^V \quad \text{and} \quad [\eta^V, \sigma^V] = 0.$$

Geometric Lagrangian Mechanics

Associated to L there is a section θ_L of $(\mathcal{T}^E E)^*$,

$$\langle \theta_L, \eta^C \rangle = d_{\eta^V} L \qquad \text{and} \qquad \langle \theta_L, \eta^V \rangle = 0.$$

Equivalent conditions:

$$i_\Gamma \omega_L = dE_L$$

with $\omega_L = -d\theta_L$ a $E_L = d_\Delta L - L$ the energy, or

$$d_\Gamma \theta_L = dL$$

with Γ a $\textsc{sode}$-section. (Martínez 2001)

A solution of Lagrange's equations is an admissible curve $a\colon \mathbb{R} \to E$ which satisfies $\delta L(\dot{a}(t)) = 0$, where

$$\langle \delta L(\dot{a}(t)), \eta(\gamma(t)) \rangle = \big(d_{\eta^C} L\big)(a(t)) - \frac{d}{dt}\big(\langle \theta_L, \eta^C \rangle (a(t))\big)$$

for every $\eta \in \mathrm{Sec}_\gamma(E)$ and where $\gamma = \tau \circ a$ is the curve on the base.

Variations $\longleftrightarrow$ Complete lifts

which are of the form

$$\begin{aligned} \delta x^i &= \rho^i_\alpha \sigma^\alpha \\ \delta y^\alpha &= \dot{\sigma}^\alpha + C^\alpha_{\beta\gamma} a^\beta \sigma^\gamma \end{aligned}$$

For an infinite-dimensional setting on the manifold of admissible curves see (Martínez 2006, ESAIM: COCV to appear)

Lie groupoids

A groupoid over a set M is a set $\boldsymbol{G}$ together with the following structural maps:

- ☐ A pair of maps (source) $\boldsymbol{\alpha}\colon \boldsymbol{G} \to M$ and (target) $\boldsymbol{\beta}\colon \boldsymbol{G} \to M$.
- ☐ A partial multiplication $\boldsymbol{m}$, defined on the set of composable pairs $\boldsymbol{G}_2 = \{\, (g,h) \in G \times G \mid \boldsymbol{\beta}(g) = \boldsymbol{\alpha}(h) \,\}$.
  - ▷ $\boldsymbol{\alpha}(gh) = \boldsymbol{\alpha}(g)$ and $\boldsymbol{\beta}(gh) = \boldsymbol{\beta}(h)$.
  - ▷ $g(hk) = (gh)k$.
- ☐ An identity section $\boldsymbol{\epsilon} : M \to \boldsymbol{G}$ such that
  - ▷ $\boldsymbol{\epsilon}(\boldsymbol{\alpha}(g))g = g$ and $g\boldsymbol{\epsilon}(\boldsymbol{\beta}(g)) = g$.
- ☐ An inversion map $\boldsymbol{i} : G \to G$, to be denoted simply by $\boldsymbol{i}(g) = g^{-1}$, such that
  - ▷ $g^{-1}g = \epsilon(\boldsymbol{\beta}(g))$ and $gg^{-1} = \boldsymbol{\epsilon}(\boldsymbol{\alpha}(g))$.

$$\begin{tikzcd} \boldsymbol{\alpha}(g) \arrow[r, bend left, "g"] & \boldsymbol{\beta}(g) \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"'] \arrow[rr, bend left, "gh"] & \bullet \arrow[r, bend left, "h"'] & \bullet \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \arrow[l, bend left, "g^{-1}"] \end{tikzcd}$$

A groupoid is a Lie groupoid is $\boldsymbol{G}$ and M are manifolds, all maps (source, target, inversion, multiplication, identity) are smooth, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are submersions (then $\boldsymbol{m}$ is a submersion, $\boldsymbol{\epsilon}$ is an embedding and $\boldsymbol{i}$ is a diffeomorphism).

The Lie algebroid of a Lie groupoid

The Lie algebroid of a Lie groupoid $\boldsymbol{G}$ is the vector bundle $\tau\colon E \to M$ where $E_m = \ker(T_{\boldsymbol{\epsilon}(m)}\boldsymbol{\alpha})$ with $\rho_m = T_{\boldsymbol{\epsilon}(m)}\boldsymbol{\beta}$.

The bracket is defined in terms of left-invariant vector fields.

Left and right translation:
$g \in \boldsymbol{G}$ with $\boldsymbol{\alpha}(g) = m$ and $\boldsymbol{\beta}(g) = n$

$$\begin{aligned} l_g &\colon \boldsymbol{\alpha}^{-1}(n) \to \boldsymbol{\alpha}^{-1}(m), \qquad & l_g(h) &= gh \\ r_g &\colon \boldsymbol{\beta}^{-1}(m) \to \boldsymbol{\beta}^{-1}(n), \qquad & r_g(h) &= hg \end{aligned}$$

Every section σ of E can be extended to a left invariant vectorfield $\overleftarrow{\sigma} \in \mathfrak{X}(\boldsymbol{G})$. The bracket of two sections of E is defined by $\overleftarrow{[\sigma, \eta]} = [\overleftarrow{\sigma}, \overleftarrow{\eta}]$.

Examples

- **Pair groupoid.**

  $G = M \times M$ with $\boldsymbol{\alpha}(m_1, m_2) = m_1$ and $\boldsymbol{\beta}(m_1, m_2) = m_2$.
  Multiplication is $(m_1, m_2)(m_2, m_3) = (m_1, m_3)$
  Identities $\boldsymbol{\epsilon}(m) = (m, m)$
  Inversion $\boldsymbol{i}(m_1, m_2) = (m_2, m_1)$.
  The Lie algebroid is $TM \to M$.

- **Lie group.**

  A Lie group is a Lie groupoid over one point $M = \{e\}$. Every pair of elements is composable.
  The Lie algebroid is just the Lie algebra.

Transformation groupoid.

Consider a Lie group H acting on a manifold M on the right. The set $\boldsymbol{G} = M \times H$ is a groupoid over M with $\boldsymbol{\alpha}(m, g) = m$ and $\boldsymbol{\beta}(m, g) = mg$.
Multiplication is $(m, h_1)(mh_1, h_2) = (m, h_1 h_2)$.
Identity $\boldsymbol{\epsilon}(m) = (m, e)$
Inversion $\boldsymbol{i}(m, h) = (mh, h^{-1})$
The Lie algebroid is the transformation Lie algebroid $M \times \mathfrak{h} \to M$.

Atiyah or gauge groupoid.

If $\pi \colon Q \to M$ is a principal H-bundle, then $(Q \times Q)/H$ is a groupoid over M, with source $\boldsymbol{\alpha}([q_1, q_2]) = \pi(q_1)$ and target $\boldsymbol{\beta}([q_1, q_2]) = \pi(q_2)$.
Multiplication is $[q_1, q_2][hq_2, q_3] = [hq_1, q_3]$.
Identity $\boldsymbol{\epsilon}(m) = [q, q]$
Inversion $\boldsymbol{i}([q_1, q_2]) = [q_2, q_1]$
(An element of $(Q \times Q)/G$ can be identified with an equivariant map between fibers)

Discrete Lagrangian Mechanics

A discrete Lagrangian on a Lie groupoid $\boldsymbol{G}$ is just a function $\mathbb{L}$ on $\boldsymbol{G}$. It defines a discrete dynamical system by mean of discrete Hamilton principle.

- Action sum: defined on composable sequences $(g_1, g_2, \cdots, g_n) \in \boldsymbol{G}_n$

$$S(g_1, g_2, \ldots, g_n) = \mathbb{L}(g_1) + \mathbb{L}(g_2) + \cdots + \mathbb{L}(g_n).$$

- Discrete Hamilton principle: Given $p \in \boldsymbol{G}$, a solution of a Lagrangian system is a critical point of the action sum on the set of composable sequences with product p, i.e. sequences $(g_1, g_2 \cdots, g_n) \in \boldsymbol{G}_n$ such that $g_1 g_2 \cdots g_n = p$

Discrete Euler-Lagrange equations

We can restrict to sequences of two elements (g, h). Since $gh = p$ is fixed, variations are of the form $g \mapsto g\eta(t)$ and $h \mapsto \eta(t)^{-1}h$, with $\eta(t)$ a curve thought the identity at $m = \boldsymbol{\beta}(g) = \boldsymbol{\alpha}(h)$ with $\dot{\eta}(0) = a \in E_m$. Then the discrete Euler-Lagrange equations are:

$$\begin{aligned} \langle D_{DEL}\mathbb{L}(g,h), a\rangle &= \frac{d}{dt}\left[\mathbb{L}(g\eta(t)) + \mathbb{L}(\eta(t)^{-1}h)\right]\Big|_{t=0} \\ &= \langle d^0\big(\mathbb{L}\circ l_g + \mathbb{L}\circ r_h \circ \boldsymbol{i}\big), a\rangle. \end{aligned}$$

Example: Heavy top

Consider the transformation Lie algebroid $\tau : S^2 \times \mathfrak{so}(3) \to S^2$ and Lagrangian

$$L_c(\Gamma, \Omega) = \frac{1}{2}\Omega \cdot I\Omega - mgl\Gamma \cdot \mathrm{e} = \frac{1}{2}\operatorname{tr}(\hat{\Omega}\mathbb{I}\hat{\Omega}^T) - mgl\Gamma \cdot \mathrm{e}.$$

where $\Omega \in \mathbb{R}^3 \simeq \mathfrak{so}(3)$ and $\mathbb{I} = \frac{1}{2}\operatorname{tr}(I)I_3 - I$.

Discretize the action by the rule

$$\hat{\Omega} = R^T\dot{R} \approx \frac{1}{h}R_k^T(R_{k+1} - R_k) = \frac{1}{h}(W_k - I_3),$$

where $W_k = R_k^T R_{k+1}$ to obtain a discrete Lagrangian (an approximation of the continuous action) on the transformation Lie groupoid $L : S^2 \times SO(3) \to \mathbb{R}$

$$L(\Gamma_k, W_k) = -\frac{1}{h}\operatorname{tr}(\mathbb{I}W_k) - hmgl\Gamma_k \cdot \mathrm{e}.$$

The value of the action on a variated sequence is

$$\begin{aligned}\lambda(t) &= L(\Gamma_k, W_k e^{tK}) + L(e^{-tK}\Gamma_{k+1}, e^{-tK}W_{k+1}) \\ &= -\frac{1}{h}\left[\mathrm{tr}(\mathbb{I}W_k e^{tK}) + mglh^2\Gamma_k \cdot \mathrm{e} + \mathrm{tr}(\mathbb{I}e^{-tK}W_{k+1}) + mglh^2(e^{-tK}\Gamma_{k+1})\right.\end{aligned}$$

where $\Gamma_{k+1} = W_k^T\Gamma_k$ (since the above pairs must be composable) and $K \in \mathfrak{so}(3)$ is arbitrary.

Taking the derivative at $t = 0$ and after some straightforward manipulations we get the DEL equations

$$M_{k+1} - W_k^T M_k W_k - mglh^2(\widehat{\Gamma_{k+1} \times} \,\mathrm{e}) = 0$$

where $M = W\mathbb{I} - \mathbb{I}W^T$.

In terms of the axial vector Π in $\mathbb{R}^3$ defined by $\hat{\Pi} = M$, we can write the equations in the form

$$\Pi_{k+1} = W_k^T\Pi_k + mglh^2\Gamma_{k+1} \times \mathrm{e}.$$

Examples

- **Pair groupoid.**

  Lagrangian: $\mathbb{L}\colon M \times M \to \mathbb{R}$ Discrete Euler-Lagrange equations:

$$D_2\mathbb{L}(x,y) + D_1\mathbb{L}(y,z) = 0.$$

- **Lie group.**

  Lagrangian: $\mathbb{L}\colon G \to \mathbb{R}$ Discrete Euler-Lagrange equations:

$$\mu_{k+1} = Ad^*_{g_k}\mu_k, \qquad \text{discrete Lie-Poisson equations}$$

where $\mu_k = r^*_{g_k} d\mathbb{L}(e)$.

- **Action Lie groupoid.**

Lagrangian: $\mathbb{L}\colon M \times H \to \mathbb{R}$ Discrete Euler-Lagrange equations: Defining $\mu_k(x, h_k) = d(\mathbb{L}_x \circ r_{h_k})(e)$, we have

$$\mu_{k+1}(xh_k, h_{k+1}) = Ad^*_{h_k}\mu_k(x, h_k) + d(L_{h_{k+1}} \circ ((xh_k)\cdot))(e),$$

where $(xh_k)\cdot : H \to M$ is the map defined by

$$(xh_k)\cdot(h) = x(h_k h).$$

These are the discrete Euler-Poincaré equations.

- **Atiyah groupoid.**

Lagrangian: $\mathbb{L}\colon (Q \times Q)/H \to \mathbb{R}$. Discrete Euler-Lagrange equations: Locally $Q = M \times H$

$$\begin{array}{l} D_2L((x,y),h_k) + D_1L((y,z),h_{k+1}) = 0, \\ \mu_{k+1}(y,z) = Ad^*_{h_k}\mu_k(x,y), \end{array} \tag{1}$$

where

$$\mu_k(\bar{x},\bar{y}) = d(r^*_{h_k}L_{(\bar{x},\bar{y},\ )})(e)$$

for $(\bar{x},\bar{y}) \in M \times M$.

One can find a global expression in terms of a discrete connection.

Simplecticity?

In the case of the pair groupoid, it is well known that the algorithm defined by the discrete Euler-Lagrange equations is symplectic.

In the general case of a Lagrangian system on a Lie groupoid one can also define a symplectic section on an appropriate Lie algebroid which is conserved by the discrete flow. From this it follows that the algorithm is Poisson (In the standard sense).

Such appropriate Lie algebroid is called the prolongation of the Lie groupoid $\mathcal{P}G \to \boldsymbol{G}$, where

$$\mathcal{P}_g\boldsymbol{G} = \ker(T_g\boldsymbol{\alpha}) \oplus \ker(T_g\boldsymbol{\beta})$$

It can be seen isomorphic to

$$\mathcal{P}\boldsymbol{G} = \{\, (a,g,b) \in E \times \boldsymbol{G} \times E \mid \tau(a) = \boldsymbol{\alpha}(g) \quad \text{and} \quad \tau(b) = \boldsymbol{\beta}(g) \,\}$$

where $\tau\colon E \to M$ is the Lie algebroid of $\boldsymbol{G}$.

Cartan forms

Given a discrete Lagrangian $\mathbb{L} \in C^\infty(\boldsymbol{G})$ we define the Cartan 1-sections $\Theta_{\mathbb{L}}^-$ and $\Theta_{\mathbb{L}}^+$ of $\mathcal{P}\boldsymbol{G}^*$ by

$$\Theta_{\mathbb{L}}^-(g)(X_g, Y_g) = -X_g(L), \qquad \text{and} \qquad \Theta_{\mathbb{L}}^+(g)(X_g, Y_g) = Y_g(L),$$

for each $g \in G$ and $(X_g, Y_g) \in V_g\beta \oplus V_g\alpha$.

The difference between them is

$$d\mathbb{L} = \Theta_{\mathbb{L}}^+ - \Theta_{\mathbb{L}}^-.$$

The Cartan 2-section is

$$\Omega_{\mathbb{L}} = -d\Theta_{\mathbb{L}}^+ = -d\Theta_{\mathbb{L}}^-$$

A Lagrangian is said to be regular if $\Omega_{\mathbb{L}}$ is a symplectic section.

Discrete evolution operator

For a regular Lagrangian there exists a locally unique map $\xi\colon \boldsymbol{G} \to \boldsymbol{G}$ such that it solves the discrete Euler-Lagrange equations

$$D_{DEL}\mathbb{L}(g, \xi(g)) = 0 \qquad \text{for all } g \text{ in an open } \mathcal{U} \subset \boldsymbol{G}.$$

One of such maps is said to be a discrete Lagrangian evolution operator.

Given a map $\xi\colon \boldsymbol{G} \to \boldsymbol{G}$ such that $\boldsymbol{\alpha} \circ \xi = \boldsymbol{\beta}$, there exists a unique vector bundle map $\mathcal{P}\xi\colon \mathcal{P}\boldsymbol{G} \to \mathcal{P}\boldsymbol{G}$, such that $\Phi = (\mathcal{P}\xi, \xi)$ is a morphism of Lie algebroids.

A map ξ is a discrete Lagrangian evolution operator if and only if

$$\Phi^*\Theta^-_{\mathbb{L}} - \Theta^-_{\mathbb{L}} = d\mathbb{L}.$$

If ξ is a discrete Lagrangian evolution operator then it is symplectic, that is, $\Phi^*\Omega_{\mathbb{L}} = \Omega_{\mathbb{L}}$.

Hamiltonian formalism

Define the discrete Legendre transformations $\mathbb{F}^{-}\mathbb{L} : \boldsymbol{G} \to E^*$ and $\mathbb{F}^{+}\mathbb{L} : \boldsymbol{G} \to E^*$ by

$$\begin{aligned} (\mathbb{F}^{-}L)(h)(a) &= -a(L \circ r_h \circ i), \qquad && \text{for } a \in E_{\boldsymbol{\alpha}(h)} \\ (\mathbb{F}^{+}L)(g)(b) &= \quad b(L \circ l_g), \qquad && \text{for } b \in E_{\boldsymbol{\beta}(g)} \end{aligned}$$

The Lagrangian is regular if and only if $\mathbb{F}^{\pm}\mathbb{L}$ is a local diffeomorphism.

If Θ is the canonical 1-section on the prolongation of E^* then

$$(\mathcal{P}\mathbb{F}^{\pm}\mathbb{L})^*\Theta = \Theta_L^{\pm},$$

and

$$(\mathcal{P}\mathbb{F}^{\pm}\mathbb{L})^*\Omega = \Omega_L.$$

We also have that

$$D_{DEL}\mathbb{L}(g,h) = \mathbb{F}^{+}L(g) - \mathbb{F}^{-}L(h)$$

so that the Hamiltonian evolution operator $\xi_{\mathbb{L}}$ is

$$\xi_{\mathbb{L}} = (\mathbb{F}^{+}L) \circ (\mathbb{F}^{-}L)^{-1},$$

which is therefore symplectic

$$(\mathcal{P}\xi_{L})^{*}\Omega = \Omega.$$

Morphisms and reduction

A morphism of Lie groupoids is a bundle map (ϕ, φ) between groupoids $\boldsymbol{G}$ over M and $\boldsymbol{G}'$ over M' such that $\Phi(gh) = \Phi(g)\Phi(h)$.

The prolongation $\mathcal{P}\phi$ of ϕ is the map $\mathcal{P}\phi(X,Y) = (T\phi(X), T\Phi(Y))$ from $\mathcal{P}\boldsymbol{G}$ to $\mathcal{P}\boldsymbol{G}'$.

Assume that we have a Lagrangian $\mathbb{L}$ on $\boldsymbol{G}$ and a Lagrangian $\mathbb{L}'$ on $\boldsymbol{G}'$ related by a morphism of Lie groupoids ϕ, that is $\mathbb{L}' \circ \phi = \mathbb{L}$. Then

- ☐ $\langle D_{DEL}\mathbb{L}(g,h), a\rangle = \langle D_{DEL}\mathbb{L}'(\phi(g), \phi(h)), \phi_*(a)\rangle$
- ☐ $\mathcal{P}\phi^*\Theta^{\pm}_{\mathbb{L}'} = \Theta^{\pm}_{\mathbb{L}}$
- ☐ $\mathcal{P}\phi^*\Omega_{\mathbb{L}'} = \Omega_{\mathbb{L}}$

As a consequence:

Let (ϕ, φ) be a morphism of Lie groupoids from $G \rightrightarrows M$ to $G' \rightrightarrows M'$ and suppose that $(g, h) \in G_2$.

1. If $(\phi(g), \phi(h))$ is a solution of the discrete Euler-Lagrange equations for $\mathbb{L}' = \mathbb{L} \circ \Phi$, then (g, h) is a solution of the discrete Euler-Lagrange equations for $\mathbb{L}$.
2. If ϕ is a submersion then (g, h) is a solution of the discrete Euler-Lagrange equations for $\mathbb{L}$ if and only if $(\phi(g), \phi(h))$ is a solution of the discrete Euler-Lagrange equations for $\mathbb{L}'$.

Examples

- Let G be a Lie group and consider the pair groupoid $G\times G$ over G. Consider also G as a groupoid over one point. Then we have that the map

  $$\begin{array}{cccc} \Phi_l : & G\times G & \longrightarrow & G \\ & (g,h) & \mapsto & g^{-1}h \end{array}$$

  is a Lie groupoid morphism, and a submersion. The discrete Euler-Lagrange equations for a left invariant discrete Lagrangian on $G\times G$ reduce to the discrete Lie-Poisson equations on G for the reduced Lagrangian.

- Let G be a Lie group acting on a manifold M by the left. We consider a discrete Lagrangian on $G \times G$ which depends on the variables of M as parameters $\mathbb{L}_m(g,h)$. The Lagrangian is invariant in the sense $\mathbb{L}_m(rg,rh) = \mathbb{L}_{r^{-1}m}(g,h)$.

  We consider the Lie groupoid $G\times G\times M$ over $G\times M$ where the elements in M as parameters, and thus $\mathbb{L} \in C^\infty(G\times G\times M)$ and then $\mathbb{L}(rg,rh,rm) = \mathbb{L}(g,h,m)$. Thus we define the reduction map (submersion)

$$\begin{array}{rccc} \Phi: & G\times G\times M & \longrightarrow & G\times M \\ & (g,h,m) & \mapsto & (g^{-1}h, g^{-1}m) \end{array}$$

  where on $G \times M$ we consider the transformation Lie groupoid defined by the right action $m \cdot g = g^{-1}m$.
  The Euler-Lagrange equations on $G\times G\times M$ reduces to the Euler-Lagrange equations on $G \times M$.

- A G-invariant Lagrangian L defined on the pair groupoid $L\colon Q\times Q\to\mathbb{R}$, where $p\colon Q\to M$ is a G-principal bundle. In this case we can reduce to the Atiyah gauge groupoid by means of the map

  $$\begin{array}{rccc}\Phi: & Q\times Q & \longrightarrow & (Q\times Q)/G\\ & (q,q') & \mapsto & [(q,q')]\end{array}$$

  Thus the discrete Euler-Lagrange equations reduce to the discrete Lagrange-Poincaré equations.

Thank you