Discrete nonholonomic Lagrangian systems on Lie groupoids

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NUMERICAL INTEGRATION OF NONHOLONOMIC SYSTEMS

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D. Iglesias, J.C. Marrero, D. Martín de Diego, E. Martínez: Discrete nonholonomic Lagrangian systems on Lie groupoids, *work in progress*

Geometric formulation of continuous constrained (nonholonomic) Lagrangian systems on Lie algebroids

- Mestdag-Langerock (2005)
- Cortés, de León, Marrero and Martínez (2005)

Ingredients

- $\tau : E \to Q$ a Lie algebroid over Q with Lie algebroid structure $([\![\cdot,\cdot]\!], \rho)$
- L:E$\to \mathbb{R}$ a regular Lagrangian function
- $\mathcal{M}$ an embedded submanifold of E such that

$$\tau_{|\mathcal{M}} : \mathcal{M} \to Q \text{ is a fibration}$$

(usually, $\mathcal{M}$ is a vector subbundle over Q of E)

$(L, \mathcal{M})$ a constrained (nonholonomic) Lagrangian system on E

Geometric formulation of continuous constrained (nonholonomic) Lagrangian systems on Lie algebroids

- (x^i) local coordinates on Q; $\{e_A\}$ a local basis of $Sec(E)$

  (x^i, y^A) the corresponding local coordinates on E

  (ρ^i_A, C^C_{AB}) the local structure functions on E

  $\rho(e_A) = \rho^i_A \frac{\partial}{\partial x^i}, \quad [\![e_A, e_B]\!] = C^C_{AB} e_C$

  $\{\Phi^\alpha(x^i, y_A) = 0,\ \alpha = 1, \ldots, s\}$ local equations defining to the submanifold $\mathcal{M}$

Continuous constrained (nonholonomic) Euler-Lagrange equations

$\gamma \equiv t \to (x^i(t), y^A(t))$ a curve on E

γ is a solution

$$\Updownarrow$$

$$\dot{x}^i = \rho^i_A y^A$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^A}\right) + \frac{\partial L}{\partial y^c} C^C_{AB} y^B - \rho^i_A \frac{\partial L}{\partial x^i} = \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial y^A}, \qquad \Phi^\alpha(x^i, y^A) = 0$$

$\lambda_\alpha \equiv$ The Lagrange multipliers

The unconstrained case: $\mathcal{M} = E$

$$\dot{x}^i = \rho^i_A y^A$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^A}\right) + \frac{\partial L}{\partial y^c} C^C_{AB} y^B - \rho^i_A \frac{\partial L}{\partial x^i} = 0$$

Martínez (2001)

- E-tangent bundle to E

$$\mathcal{T}^E E = \{(e, X) \in E \times TE / \rho(e) = (T\tau)(X)\}$$

a Lie algebroid over E

- The vertical endomorphism

$$S(e, X) = (0, e^v_{e'}), \quad (e, X) \in \mathcal{T}^E_{e'} E$$

- The Poincaré-Cartán 2-section associated with L

$$\omega_L = -d^{\mathcal{T}^E E}(S^*(d^{\mathcal{T}^E E} L)) \in Sec(\wedge^2(\mathcal{T}^E E)^*)$$

- The Lagrangian energy

$$E_L = \Delta(L) - L \in C^\infty(E)$$

L regular $\Rightarrow \omega_L$ is a symplectic section

$$\flat_{\omega_L} : \mathcal{T}^E E \to (\mathcal{T}^E E)^* \text{ the musical isomorphism}$$

- E-tangent bundle to $\mathcal{M}$

$$\mathcal{T}^E\mathcal{M} = \{(e, Y) \in E \times T\mathcal{M} / \rho(e) = (T\tau_M)(Y)\}$$

- Geometric formulation of the continuous constrained Euler-Lagrange equations for the constrained system $(L, \mathcal{M})$

$$\Gamma \in Sec(\mathcal{T}^E E)$$

$$(i_\Gamma \omega_L - d^{\mathcal{T}^E E} E_{)|\mathcal{M}} \in Sec(S^*(\mathcal{T}^E M)^0)$$

$$\Gamma_{|M} \in Sec(\mathcal{T}^E E)$$

$(L, \mathcal{M})$ regular if the above equations admit a unique solution

$F, \mathcal{H}$ vector subbundles of $\mathcal{T}^E E$ over $\mathcal{M}$

$$a \in \mathcal{M} \to F_a^0 = S^*(\mathcal{T}_a^E \mathcal{M})^0 \subseteq (\mathcal{T}_a^E E)^*$$

$$a \in \mathcal{M} \to \mathcal{H}_a = \{z \in \mathcal{T}_a^E \mathcal{M} / S(z) \in \mathcal{T}_a^E \mathcal{M}\}$$

- $(L, \mathcal{M})$ is a regular constrained Lagrangian system

$$\Updownarrow$$

$$\mathcal{T}^E \mathcal{M} \cap F^\perp = \{0\}$$

$$\Updownarrow$$

$\mathcal{H}$ is a symplectic subundle of the symplectic bundle $(\mathcal{T}_{\mathcal{M}}^E E, \omega_L)$

"One of the main features of the Lie algebroid framework is its inclusive nature"

Continuous constrained Lagrangian system	Lie algebroid	Dynamical equations	Examples
Standard nonholonomic Lagrangian systems	TQ	Standard nonholonomic E-L equations	All the constrained systems with Lagrangian function on TQ [1]
Nonholonomic Lagrangian systems on Lie algebras	$\mathfrak{g} \equiv$ a real Lie algebra	Nonholonomic Euler -Poincaré equations	Suslov system ($\mathfrak{g} = \mathfrak{so}(3)$) [2]
Nonholonomic Mechanical systems with configuration space a compact Lie group which are invariant under the action of a closed Lie subgroup	An action Lie algebroid	Poincaré-Chetayev (or Bolzano-Hamel) equations	LR-systems (Veselova problem or multidimensional generalizations of this problem) [3]
Non-holonomic Lagrangian systems which are invariant under the action of a symmetry Lie group G	An Atiyah (gauge) algebroid associated with a principal G-bundle	Nonholonomic Lagrange-Poincaré equations	A ball rolling on a rotating table; A two wheeled planar mobile robot etc....[4]

[1] Monographs by Bloch (2003) and Cortés (2002); Survey Paper by Cendra, Marsden, Ratiu (2001)
[2] Fedorov-Kozlov (1995), Jovanovic (1999), Zenkov-Bloch (2000)
[3] Veselov-Veselova (1986,1988), Fedorov-Jovanovic (2004)
[4] Monograph by Cortes (2002)

Lie algebroids$\equiv$ Infinitesimal invariants of Lie groupoids

Geometric formulation of discrete unconstrained Lagrangian Mechanics on Lie groupoids

Marrero, Martín de Diego and Martínez (2006)
(the topic of the previous talk)

All the above facts provide a good motivation for developing a generalized theory of discrete constrained (nonholonomic) Lagrangian Mechanics on Lie groupoids

Some steps have been given in this direction:
Cortés, Martínez (2001): Pair groupoid
Federov, Zenkov (2005): Lie groups
McLachlan, Perlmuther (2006): Lie groups

Plain of the talk

1. Discrete generalized Hölder's Principle
2. Discrete nonholonomic Legendre transformations
3. Nonholonomic evolution operators and regular discrete nonholonomic Lagrangian systems
4. Reversible discrete nonholonomic Lagrangian systems
5. Lie groupoid morphisms and discrete nonholonomic dynamics
6. The discrete nonholonomic momentum map
7. Some examples

Discrete generalized Holder's principle

$\Gamma \rightrightarrows Q$ a Lie groupoid; $\dim \Gamma = m+n$, $\dim Q = m$

$\alpha, \beta : \Gamma \to Q$, $\epsilon : Q \to \Gamma; i : \Gamma \to \Gamma$, $m : \Gamma_2 \to \Gamma$

$\tau : E_\Gamma \to Q \equiv$ the Lie algebroid of Γ

Generalized discrete nonholonomic (or constrained) Lagrangian system

- $L_d : \Gamma \to \mathbb{R}$ a regular discrete Lagrangian
- The constraint distribution $\mathcal{D}_c$

  $\tau_{\mathcal{D}_c} : \mathcal{D}_c \to Q$ a vector subbundle of E_Γ, $rank \mathcal{D}_c = r$
- The discrete constraint embedded submanifold $\mathcal{M}_c$

  $i_{\mathcal{M}_c} : \mathcal{M}_c \to \Gamma$ is a embedded submanifold of Γ

Assumption

$$\dim \mathcal{M}_c = \dim \mathcal{D}_c = m + r, \quad r \leq n$$

$(L_d, \mathcal{M}_c, \mathcal{D}_c) \equiv$ a discrete nonholonomic Lagrangian system on Γ

$g \in \Gamma$ fixed

$$\mathcal{C}_g^N = \{(g_1, \ldots, g_N) \in \Gamma^N / (g_k, g_{k+1}) \in \Gamma_2, \text{ for } k = 1, .., N-1 \text{ and } g_1 \ldots g_N = g\}$$

$$T_{(g_1, g_2, .., g_N)}\mathcal{C}_g^N \equiv \{(v_1, v_2, \ldots, v_{N-1}) \mid v_k \in (E_\Gamma)_{x_k} \text{ and } x_k = \beta(g_k), 1 \leq k \leq N-1\}$$

discrete action sum

$$\mathcal{S}L_d : \mathcal{C}_g^N \longrightarrow \mathbb{R} \quad (g_1, \ldots, g_N) \longmapsto \sum_{k=1}^{N} L_d(g_k)$$

$$(\mathcal{V}_c)_{(g_1, \ldots, g_N)} = \{(v_1, \ldots, v_{N-1}) \in T_{(g_1, \ldots, g_N)}\mathcal{C}_g^N / \forall k \in \{1, \ldots, N-1\}, \; v_k \in \mathcal{D}_c\}$$

Discrete Hölder's principle

$g \in \Gamma, \quad (g_1, \ldots, g_N) \in \mathcal{C}_g^N$

$(g_1, \ldots, g_N)$ is a solution of the discrete nonholonomic Lagrangian system

$$(L_d, \mathcal{M}_c, \mathcal{D}_c)$$

$$\Updownarrow$$

- $g_k \in \mathcal{M}_c, \quad \forall k \in \{1, \ldots, N\}$
- $\delta \mathcal{S}L_d|_{(\mathcal{V}_c)_{g_1 \ldots, g_N}} = 0$

$$(g_1, \ldots, g_N) \in \mathcal{C}_g^N$$

$$\Downarrow$$

- $g_k \in \mathcal{M}_c, \quad \forall k \in \{1, \ldots, N\}$
- $\displaystyle\sum_{k=1}^{N-1} (d^o(L_d \circ l_{g_k}) + d^o(L_d \circ r_{g_{k+1}} \circ i))(\epsilon(\beta(g_k))_{|(\mathcal{D}_c)(\beta(g_k))} = 0$

$\beta(g_k) = \alpha(g_{k+1}) = x_k$

$N = 2, \quad (g, h) \in \Gamma_2, \quad \beta(g) = \alpha(h) = x$

$$(g, h) \text{ is a solution}$$

$$\Updownarrow$$

$$(g, h) \in \mathcal{M}_c \times \mathcal{M}_c, \qquad d^o(L_d \circ l_g + L_d \circ r_h \circ i)(\epsilon(x))_{|(\mathcal{D}_c)_x} = 0$$

Discrete nonholonomic Euler-Lagrange equations for the system $(L_d, \mathcal{M}_c, \mathcal{D}_c)$

Particular case:The system is unconstrained

$$(\mathcal{M}_c = \Gamma, \mathcal{D}_c = E_\Gamma)$$

$$\Downarrow$$

$$d^0(L_d \circ l_g + L_d \circ r_h \circ i)(\epsilon(x)) = 0$$

Discrete Euler Lagrange Equations for the Lagrangian L_d

Alternative versions of the discrete nonholonomic E-L equations

$$\{X^\alpha\} \text{ a local basis of } \Gamma(\mathcal{D}_c^0)$$

$$(g,h) \in \Gamma_2, \quad \beta(g) = \alpha(h) = x \in Q$$

$$(g,h) \text{ is a solution}$$

$$\Updownarrow$$

$$(g,h) \in \mathcal{M}_c \times \mathcal{M}_c$$

$$d^o\left[L_d \circ l_g + L_d \circ r_h \circ i\right](\epsilon(x))(v) = \lambda_\alpha X^\alpha(x)(v),$$

$$\lambda_\alpha \equiv \text{ the Lagrange multipliers}$$

$$\Gamma = Q \times Q$$

$$(q_0, q_1) \in \mathcal{M}_c$$

$$((q_0,q_1),(q_1,q_2)) \text{ is a solution}$$

$$\Updownarrow$$

$$(q_1, q_2) \in \mathcal{M}_c$$

$$D_2 L_d(q_0,q_1) + D_1 L_d(q_1,q_2) = \lambda_\alpha A^\alpha(q_1)$$

Cortés, Martínez (2001)

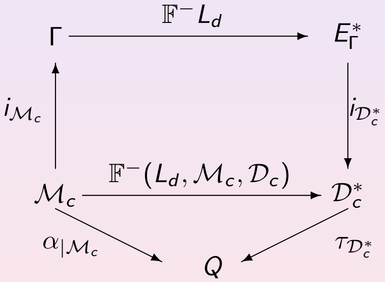
McLachlan, Perlmuther(2006)

Discrete nonholonomic Legendre transformations

$(L_d, \mathcal{M}_c, \mathcal{D}_c)$ a discrete nonholonomic Lagrangian system on Γ

The (minus) discrete nonholonomic Legendre transformation

$$\mathbb{F}^-(L_d, \mathcal{M}_c, \mathcal{D}_c) : \mathcal{M}_c \to \mathcal{D}_c^*$$



$$\mathbb{F}^-(L_d, \mathcal{M}_c, \mathcal{D}_c)(h)(v_{\epsilon(\alpha(h))}) = -v_{\epsilon(\alpha(h))}(L_d \circ r_h \circ i)$$

$v_{\epsilon(\alpha(h))} \in \mathcal{D}_c(\alpha(h))$

- The symplectic Lie algebroid $\tilde{\tau}_\Gamma : \mathcal{T}^\Gamma\Gamma \to \Gamma$

$$\mathcal{T}^\Gamma\Gamma = V\beta \oplus_\Gamma V\alpha \quad \Gamma\text{-tangent bundle to } \Gamma$$

$$([\![\cdot,\cdot]\!]^{\mathcal{T}^\Gamma\Gamma}, \rho^{\mathcal{T}^\Gamma\Gamma}) \text{ the Lie algebroid structure}$$

$$L_d : \Gamma \to \mathbb{R} \text{ regular} \Rightarrow \begin{array}{l}\text{The Poincaré-Cartan 2-section}\\ \Omega_{L_d} \text{ is symplectic}\end{array}$$

- $h \in \Gamma \Rightarrow {}_h^{(1,0)} : (E_\Gamma)^*_{\alpha(h)} \to (\mathcal{T}_h^\Gamma\Gamma)^* \equiv V_h^*\beta \oplus V_h^*\alpha$ a linear monomorphism

$$\gamma \in (E_\Gamma)^*_{\alpha(h)} \Rightarrow \gamma_h^{(1,0)}(X_h, Y_h) = \gamma(T_h(i \circ r_h^{-1})(X_h)),$$
$$(X_h, Y_h) \in \mathcal{T}_g^\Gamma\Gamma \equiv V_h\beta \oplus V_h\alpha$$

- F a vector subbundle (of rank $n + r$) of $\tilde{\tau}^\Gamma : \mathcal{T}^\Gamma\Gamma \to \Gamma$

$$h \in \Gamma \Rightarrow F_h^0 = \{\gamma_h^{(1,0)} / \gamma \in \mathcal{D}_c(\alpha(h))^0\} \subseteq (\mathcal{T}_h^\Gamma\Gamma)^*$$

- $h \in \Gamma \Rightarrow \mathcal{H}_h = (\rho^{\mathcal{T}^\Gamma\Gamma})^{-1}(T_h\mathcal{M}_c) \cap F_h \subseteq \mathcal{T}^\Gamma\Gamma$

Theorem

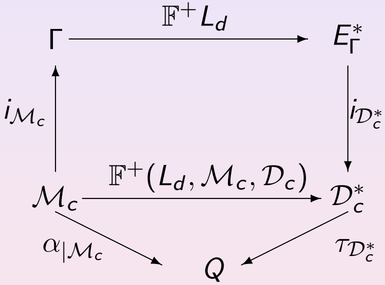
Let $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ be a discrete nonholonomic Lagrangian system. Then, the following conditions are equivalent:

1. The discrete nonholonomic Legendre transformation $\mathbb{F}^{-}(L_d, \mathcal{M}_c, \mathcal{D}_c)$ is a local diffeomorphism.
2. For every $h \in \mathcal{M}_c$
$$(\rho^{\mathcal{T}^{\Gamma}\Gamma})^{-1}(T_h\mathcal{M}_c) \cap F_h^{\perp} = \{0\}$$
3. For every $h \in \mathcal{M}_c$ the dimension of the vector subspace $\mathcal{H}_h$ is $2r$ and the restriction to the vector subbundle of the Poincaré-Cartan 2-section Ω_{L_d} is nondegenerate

$$F_h^{\perp} = \{(X_h, Y_h) \in \mathcal{T}^{\Gamma}\Gamma / \Omega_{L_d}(h)((X_h, Y_h), (X'_h, Y'_h)) = 0, \ \forall (X'_h, Y'_h) \in F_h\}$$

The (plus) discrete nonholonomic Legendre transformation

$$\mathbb{F}^+(L_d, \mathcal{M}_c, \mathcal{D}_c) : \mathcal{M}_c \to \mathcal{D}_c^*$$



$$\mathbb{F}^+(L_d, \mathcal{M}_c, \mathcal{D}_c)(g)(v_{\epsilon(\beta(g))}) = v_{\epsilon(\beta(g))}(L_d \circ l_g), \quad v_{\epsilon(\beta(g))} \in \mathcal{D}_c(\beta(g)$$

- $g \in \Gamma \Rightarrow {}_g^{(1,0)} : (E_\Gamma)^*_{\beta(g)} \to (\mathcal{T}^\Gamma_g \Gamma)^* \equiv V_g^* \beta \oplus V_g^* \alpha$ a linear monomorphism.

$$\Gamma \in (E_\Gamma)^*_{\beta(g)} \Rightarrow \gamma_g^{(1,0)}(X_g, Y_g) = \gamma((T_g l_{g^{-1}})(Y_g))$$

$$(X_g, Y_g) \in \mathcal{T}^\Gamma_g \Gamma \equiv V_g \beta \oplus V_g \alpha$$

- $\bar{F}$ a vector subbundle (of rank $n + r$) of $\tilde{\tau}_\Gamma : \mathcal{T}^\Gamma \Gamma \to \Gamma$

$$g \in \Gamma \Rightarrow \bar{F}_g^0 = \{\gamma_g^{(1,0)} / \gamma \in \mathcal{D}_c(\beta(g))^0\} \subseteq (T_g^\Gamma \Gamma)^*$$

- $g \in \Gamma \Rightarrow \bar{\mathcal{H}} g = (\rho^{\mathcal{T}^\Gamma \Gamma})^{-1}(T_g \mathcal{M}_c) \cap \bar{F}_g \subseteq \mathcal{T}_g^\Gamma \Gamma$

Theorem

Let $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ be a discrete nonholonomic Lagrangian system. Then, the following conditions are equivalent:

1. The discrete nonholonomic Legendre transformation $\mathbb{F}^{+}(L_d, \mathcal{M}_c, \mathcal{D}_c)$ is a local diffeomorphism.
2. For every $g \in \mathcal{M}_c$
$$(\rho^{\mathcal{T}^\Gamma\Gamma})^{-1}(T_g\mathcal{M}_c) \cap \bar{F}_g^{\perp} = \{0\}$$
3. For every $g \in \mathcal{M}_c$ the dimension of the vector subspace $\bar{\mathcal{H}}_g$ is $2r$ and the restriction to the vector subbundle $\bar{\mathcal{H}}$ of the Poincaré-Cartan 2-section Ω_{L_d} is nondegenerate.

$$F_g^{\perp} = \{(X_g, Y_g) \in \mathcal{T}_g^{\Gamma}\Gamma / \Omega_{L_d}(g)((X_g, Y_g), (X'_g, Y'_g)) = 0, \forall (X'_g, Y'_g) \in \bar{F}_g\}$$

Nonholonomic evolution operators and regular discrete nonholonomic Lagrangian systems

$(L_d, \mathcal{M}_c, \mathcal{D}_c)$ a discrete nonholonomic Lagrangian system on Γ

$\Upsilon_{nh} : \mathcal{M}_c \to \mathcal{M}_c$ an smooth map

Υ_{nh} is a discrete nonholonomic evolution operator for $(L_d, \mathcal{M}_c, \mathcal{D}_c)$

$$\Updownarrow$$

- $(g, \Upsilon_{nh}(g)) \in \Gamma_2, \quad \forall g \in \mathcal{M}_c$
- $(g, \Upsilon_{nh}(g))$ is a solution of the discrete nonholonomic equations

$$d^o(L_d \circ l_g + L_d \circ r_{\Upsilon_{nh}(g)} \circ i)(\epsilon(\beta(g)))_{|\mathcal{D}_c(\beta(g))} = 0, \quad \forall g \in \mathcal{M}_c$$

$$\Updownarrow$$

$$\mathbb{F}^-(L_d, \mathcal{M}_c, \mathcal{D}_c) \circ \Upsilon_{nh} = \mathbb{F}^+(L_d, \mathcal{M}_c, \mathcal{D}_c)$$

$(L_d, \mathcal{M}_c, \mathcal{D}_c)$ is said to be a regular discrete nonholonomic Lagrangian system

$$\Updownarrow$$

$\mathbb{F}^{-}(L_d, \mathcal{M}_c, \mathcal{D}_c)$ and $\mathbb{F}^{+}(L_d, \mathcal{M}_c, \mathcal{D}_c)$ are local diffeomorphisms

Corollary

Let $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ be a discrete nonholonomic Lagrangian system. Then, the following conditions are equivalent:

1. The system $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ is regular.
2. The following relations hold

$$(\rho^{\mathcal{T}^{\Gamma}\Gamma})^{-1}(T_h\mathcal{M}_c) \cap F_h^{\perp} = \{0\}, \text{ for all } h \in \mathcal{M}_c,$$

$$(\rho^{\mathcal{T}^{\Gamma}\Gamma})^{-1}(T_g\mathcal{M}_c) \cap \bar{F}_g^{\perp} = \{0\}, \text{ for all } g \in \mathcal{M}_c.$$

3. $\mathcal{H}$ and $\bar{\mathcal{H}}$ are symplectic subbundles of rank $2r$ of the sympletic vector bundle $(\mathcal{T}^{\Gamma}_{\mathcal{M}_c}\Gamma, \Omega_{L_d})$.

Theorem

Let $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ be a regular discrete nonholonomic Lagrangian system and $(g_0, h_0) \in \mathcal{M}_c \times \mathcal{M}_c$ be a solution of the discrete nonholonomic equations for $(L_d, \mathcal{M}_c, \mathcal{D}_c)$. Then, there exist two open subsets U_0 and V_0 of Γ, with $g_0 \in U_0$ and $h_0 \in V_0$, and there exists a local discrete nonholonomic evolution operator $\Upsilon_{nh}^{(L_d,\mathcal{M}_c,\mathcal{D}_c)} : U_0 \cap \mathcal{M}_c \to V_0 \cap \mathcal{M}_c$ such that:

1. $\Upsilon_{nh}^{(L_d,\mathcal{M}_c,\mathcal{D}_c)}(g_0) = h_0$;
2. $\Upsilon_{nh}^{(L_d,\mathcal{M}_c,\mathcal{D}_c)}$ is a diffeomorphism and
3. $\Upsilon_{nh}^{(L_d,\mathcal{M}_c,\mathcal{D}_c)}$ is unique, that is, if U_0' is an open subset of Γ, with $g_0 \in U_0'$, and $\Upsilon_{nh} : U_0' \cap \mathcal{M}_c \to \mathcal{M}_c$ is a (local) discrete nonholonomic evolution operator then

$$(\Upsilon_{nh}^{(L_d,\mathcal{M}_c,\mathcal{D}_c)})_{|U_0 \cap U_0' \cap \mathcal{M}_c} = (\Upsilon_{nh})_{|U_0 \cap U_0' \cap \mathcal{M}_c}$$

$$(\Upsilon_{nh}^{(L_d,\mathcal{M}_c,\mathcal{D}_c)}) = (\mathbb{F}^-(L_d, \mathcal{M}_c, \mathcal{D}_c)^{-1} \circ \mathbb{F}^+(L_d, \mathcal{M}_c, \mathcal{D}_c))_{|U_0 \cap \mathcal{M}_c}$$

Reversible discrete nonholonomic Lagrangian systems

$(L_d, \mathcal{M}_c, D_c)$ a discrete nonholonomic Lagrangian system on Γ

$(L_d, \mathcal{M}_c, \mathcal{D}_c)$ is said to be reversible if

$$L_d \circ i = L_d, \quad i(\mathcal{M}_c) = \mathcal{M}_c$$

$$(L_d, \mathcal{M}_c, \mathcal{D}_c) \text{ is reversible}$$
$$\Downarrow$$
$$\mathbb{F}^+(L_d, \mathcal{M}_c, \mathcal{D}_c) = -\mathbb{F}^-(L_d, \mathcal{M}_c, \mathcal{D}_c) \circ i$$

Corollary

Let $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ be a reversible nonholonomic Lagrangian system on a Lie groupoid Γ. Then, the following conditions are equivalent:

1. The discrete nonholonomic Legendre transformation $\mathbb{F}^-(L_d, \mathcal{M}_c, \mathcal{D}_c)$ is a local diffeomorphism.
2. The discrete nonholonomic Legendre transformation $\mathbb{F}^+(L_d, \mathcal{M}_c, \mathcal{D}_c)$ is a local diffeomorphism.

Proposition

Let $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ be a reversible nonholonomic Lagrangian system on a Lie groupoid Γ and (g, h) be a solution of the discrete nonholonomic Euler-Lagrange equations for $(L_d, \mathcal{M}_c, \mathcal{D}_c)$. Then, (h^{-1}, g^{-1}) is also a solution of these equations. In particular, if the system $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ is regular and $\Upsilon_{nh}^{(L_d,\mathcal{M}_c,\mathcal{D}_c)}$ is the (local) discrete nonholonomic evolution operator for $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ then $\Upsilon_{nh}^{(L_d,\mathcal{M}_c,\mathcal{D}_c)}$ is reversible, that is,

$$\Upsilon_{nh}^{(L_d,\mathcal{M}_c,\mathcal{D}_c)} \circ i \circ \Upsilon_{nh}^{(L_d,\mathcal{M}_c,\mathcal{D}_c)} = i.$$

Lie groupoid morphims and discrete nonhomolomic dynamics

$$\begin{array}{ccc} \Gamma & \xrightarrow{\quad \Phi \quad} & \Gamma' \\ \downdownarrows & & \downdownarrows \\ Q & \xrightarrow{\quad \Phi_0 \quad} & Q' \end{array}$$

a Lie groupoid morphism

$$\begin{array}{ccc} E_\Gamma & \xrightarrow{\quad E(\Phi) \quad} & E_{\Gamma'} \\ \downarrow & & \downarrow \\ Q & \xrightarrow{\quad \Phi_0 \quad} & Q' \end{array}$$

the corresponding Lie algebroid morphism

Lie groupoid morphims and reduction

$(L_d, \mathcal{M}_c, \mathcal{D}_c), (L'_d, \mathcal{M}'_c, \mathcal{D}'_c)$ discrete nonholonomic Lagrangian systems on Γ and Γ'

Proposition

- $L_d = L'_d \circ \Phi$
- $(g, h) \in \Gamma_2 \cap (\mathcal{M}_c \times \mathcal{M}_c) / (\Phi(g), \Phi(h)) \in \mathcal{M}'_c \times \mathcal{M}'_c$
- $(E_{\beta(g)}(\Phi))(\mathcal{D}_c(\beta(g))) = (\mathcal{D}'_c)\Phi_0(\beta(g))$

$$\Downarrow$$

(g, h) is a solution for $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ if and only if $(\Phi(g), \Phi(h))$ is a solution for $(L'_d, \mathcal{M}'_c, \mathcal{D}'_c)$

The discrete nonholonomic momentum map

- $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ a regular discrete nonholonomic

$$\text{Lagrangian system on } \Gamma \rightrightarrows Q$$

$\tau : E_\Gamma \to Q$ the Lie algebroid of Γ

- $\mathfrak{g}$ a real Lie algebra of finite dimension

$$\Psi : \mathfrak{g} \to Sec(\tau) \text{ a } \mathbb{R}\text{-linear map}$$

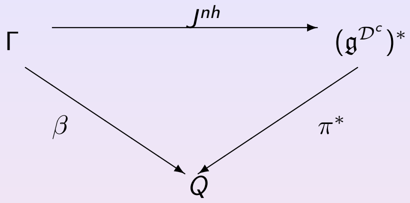
$$x \in Q \Rightarrow \mathfrak{g}^x = \{\xi \in \mathfrak{g} / \Psi(\xi)(x) \in (\mathcal{D}_c)_x\}$$

$$\mathfrak{g}^{\mathcal{D}_c} = \bigcup_{x \in Q} \mathfrak{g}^x \quad (\mathfrak{g}^{\mathcal{D}_c *}) = \bigcup_{x \in Q} (\mathfrak{g}^x)^*$$

$$\pi : \mathfrak{g}^{\mathcal{D}_c} \to Q$$

$$\pi^* : (\mathfrak{g}^{\mathcal{D}_c})^* \to Q \text{ the canonical projections}$$

The discrete nonholonomic momentum map



$$J^{nh}(g)(\xi) = \overleftarrow{\Psi(\xi)}(g)(L_d), \quad g \in \Gamma, \quad \xi \in \mathfrak{g}^{\beta(g)}$$

$$\widetilde{\xi} : Q \longrightarrow \mathfrak{g} \text{ an smooth map } / \widetilde{\xi}(x) \in \mathfrak{g}^x, \ \ \forall x \in Q$$

$$\Downarrow$$

$$J^{nh}_{\widetilde{\xi}} : \Gamma \longrightarrow \mathbb{R} \text{ an smooth function}$$

$$J^{nh}_{\widetilde{\xi}}(g) = J^{nh}(g)(\widetilde{\xi}(\beta(g))), \quad \forall g \in \Gamma$$

L_d is said to be $\mathfrak{g}$ invariant with respect to Ψ if

$$\overleftarrow{\Psi(\xi)}(L_d) - \overrightarrow{\psi(\xi)}(L_d) = 0, \quad \forall \xi \in \mathfrak{g}$$

Theorem

Let $\Upsilon_{nh}^{(L_d,\mathcal{M}_c,\mathcal{D}_c)} : \mathcal{M}_c \longrightarrow \mathcal{M}_c$ be the local discrete nonholonomic evolution operator for the system $(L_d, \mathcal{M}_c, \mathcal{D}_c)$. If L_d is $\mathfrak{g}$-invariant with respect to $\Psi : \mathfrak{g} \longrightarrow Sec(\tau)$ and $\widetilde{\xi} : M \longrightarrow \mathfrak{g}$ is a smooth map such that $\widetilde{\xi}(x) \in (\mathcal{D}_c)_x$, for all $x \in M$, then

$$\begin{aligned} J_{\widetilde{\xi}}^{nh}(\Upsilon_{nh}^{(L_d,\mathcal{M}_c,\mathcal{D}_c)}(g)) - J_{\xi}^{nh}(g) \quad &= \\ &= \overleftarrow{\Psi(\widetilde{\xi}(\beta(\Upsilon_{nh}^{(L_d,\mathcal{M}_c,\mathcal{D}_c)}(g))) - \widetilde{\xi}(\beta(g)))}(\Upsilon_{nh}^{(L_d,\mathcal{M}_c,\mathcal{D}_c)}(g))(L_d) \end{aligned}$$

for $g \in \mathcal{M}_c$

The discrete nonholonomic momentum equation

$\xi \in \mathfrak{g}$ is a horizontal symmetry for the system $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ and the map $\Psi : \mathfrak{g} \to Sec(\tau)$

$$\Updownarrow$$

$$\Psi(\xi)(x) \in (\mathcal{D}_c)_x, \quad \forall x \in Q$$

Corollary

If L_d is $\mathfrak{g}$-invariant with respect to Ψ and $\xi \in \mathfrak{g}$ is a horizontal symmetry for $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ and $\Psi : \mathfrak{g} \to Sec(\tau)$ then $J_\xi^{\mathrm{nh}} : \Gamma \to \mathbb{R}$ is a constant of the motion for $\Upsilon_{nh}^{(L_d, \mathcal{M}_c, \mathcal{D}_c)}$, that is,

$$J_\xi^{nh} \circ \Upsilon_{nh}^{(L_d, \mathcal{M}_c, \mathcal{D}_c)} = J_\xi^{nh}.$$

Some Examples

Discrete nonholonomic Lagrangian systems on Lie groups

G Lie group with Lie algebra $\mathfrak{g}$

$$\Downarrow$$

G a Lie groupoid over a single point and $\mathfrak{g}$ is the Lie algebroid associated with G

$(L_d, \mathcal{M}_c, \mathcal{D}_c)$ a discrete nonholonomic Lagrangian system on the Lie group G:

- $L_d : G \longrightarrow \mathbb{R}$ a discrete Lagrangian
- $\mathcal{M}_c$ a submanifold of G
- $\mathcal{D}_c$ a vector subspace of $\mathfrak{g}$

$$g_1 \in \mathcal{M}_c$$

$(g_1, g_2) \in G \times G$ is a solution of the discrete nonholonomic Euler-Lagrange equations for $(L_d, \mathcal{M}_c, \mathcal{D}_c)$

$$\Updownarrow$$

$$g_1^{-1} dL_d(g_1) - dL_d(g_2) g_2^{-1} = \sum_{j=1}^{n-r} \lambda^j \mu_j,$$
$$g_1 \in \mathcal{M}_c$$

λ^j the Lagrange multipliers
$\{\mu_j\}$ a basis of $\mathcal{D}_c^0$

Notation: $g, h \in G$, $\alpha_h \in T_h^* G$

$$g\alpha_h = (T_{gh}^* l_{g^{-1}})(\alpha_h) \in T_{gh}^* G, \quad \alpha_h g = (T_{hg}^* r_{g^{-1}})(\alpha_h) \in T_{hg}^* G.$$

Federov, Zenkov (2005)

McLachLan, Perlmutter (2006)

Some Examples

An example of a discrete nonoholonomic Lagrangian systems on an Atiyah Lie groupoid

"A (homogeneous) sphere of radius $r > 0$, mass m and inertia about any axis I rolls without sliding on a horizontal table which relates with constant angular velocity Ω about a vertical axis through one of its points"

- Configuration space

$$Q = \mathbb{R}^2 \times SO(3), \qquad (x, y; R) \in Q$$

- The Lagrangian function

$$L : TQ \to \mathbb{R}$$

$$L(x, y; R, \dot{x}, \dot{y}; \dot{R}) = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{4} I tr(\dot{R}R^T(\dot{R}R^T)^T)$$

- The constrained submanifold $\mathcal{M}$

$$\mathcal{M} = \{(x, y; R, \dot{x}, \dot{y}; \dot{R}) / \qquad \begin{array}{l} \dot{x} + \frac{r}{2} tr(\dot{R}R^T E_L) = -\Omega y \\ \dot{y} - \frac{r}{2} tr(\dot{R}R^T E_1) = \Omega x \} \end{array}$$

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The constrained system is $SO(3)$-invariant

$$\Downarrow$$

$(L', \mathcal{M}')$ a constrained Lagrangian system on the corresponding Atiyah algebroid

$$E' \cong TQ/SO(3) \to \mathbb{R}^2$$

- The vector bundle $E' = TQ/SO(3) \cong T\mathbb{R}^2 \times \mathfrak{so}(3) \to \mathbb{R}^2$
- The anchor map $\rho' : E' \cong T\mathbb{R}^2 \times \mathfrak{so}(3) \to T\mathbb{R}^2$ the projection over the first factor
- The Lie bracket on $Sec(E')$

$$[\![s_3', s_4']\!]' = s_5', \quad [\![s_4', s_5']\!]' = s_3', \quad [\![s_5', s_3']\!]' = s_4',$$

$\{s_i'\}_{i=1,\dots,5}$ the canonical basis of $Sec(E')$

- The reduced Lagrangian function:

$$L'(x, y, \dot{x}, \dot{y}; w) = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{4} Itr(w^2)$$

- The reduced constraint submanifold

$$\mathcal{M}' = \{(x, y, \dot{x}, \dot{y}; w) / \dot{x} + \frac{r}{2} tr(wE_2) = -\Omega y, \dot{y} - \frac{r}{2} tr(wE_1) = \Omega x\}$$

Objective

To discretize the nonholonomic Lagrangian system $(L', \mathcal{M}')$ on the Atiyah algebroid $E' \cong T\mathbb{R}^2 \times \mathfrak{so}(3) \longrightarrow \mathbb{R}^2$

The discrete Atiyah groupoid

$\Gamma' \cong \mathbb{R}^2 \times \mathbb{R}^2 \times SO(3) \rightrightarrows \mathbb{R}^2$

The discrete Lagrangian function

Using an approximation of the (local) inverse of the exponential map

$$exp : \mathfrak{so}(3) \longrightarrow SO(3)$$
$$\Downarrow$$
$$L'_d(x_0, y_0, x_1, y_1; W_1) = \frac{1}{2} m[(\frac{x_1 - x_0}{h})^2 + (\frac{y_1 - y_0}{h})^2] + \frac{I}{(2h)^2} tr(W_1)$$

Some examples

The discrete constraint submanifold $\mathcal{M}'_c$

$$\begin{aligned}
\frac{x_1 - x_0}{h} + \frac{r}{2h}\mathrm{tr}(W_1 E_2) &= -\Omega\frac{y_1 + y_0}{2}, \\
\frac{y_1 - y_0}{h} - \frac{r}{2h}\mathrm{tr}(W_1 E_1) &= \Omega\frac{x_1 + x_0}{2},
\end{aligned}$$

The discrete constraint distribution

$$\mathcal{D}'_c = < \{s'_5, rs'_1 + s'_4, rs'_2 - s'_3\} >$$

The discrete constrained Euler-Lagrange equations for $(L_d', \mathcal{M}_c', \mathcal{D}_c')$

Lie algebroid morphism

$$
\begin{aligned}
\frac{x_2 - 2x_1 + x_0}{h^2} + \frac{I\Omega}{I + mr^2}\frac{y_2 - y_0}{2h} &= 0 \\
\frac{y_2 - 2y_1 + y_0}{h^2} - \frac{I\Omega}{I + mr^2}\frac{x_2 - x_0}{2h} &= 0 \\
\mathrm{tr}\left((W_1 - W_2)E_3\right) &= 0 \\
\frac{x_2 - x_1}{h} + \frac{r}{2h}\mathrm{tr}(W_2 E_2) + \Omega\frac{y_2 + y_1}{2} &= 0, \\
\frac{y_2 - y_1}{h} - \frac{r}{2h}\mathrm{tr}(W_2 E_1) - \Omega\frac{x_2 + x_1}{2} &= 0
\end{aligned}
$$

$(x_0, y_0, x_1, y_1; W_1)$ are known