An introduction to nonholonomic mechanics and its applications

Hernán Cendra

Departamento de Matemática
Universidad Nacional del Sur

December 21, 2006

Table of Contents

Nonholonomic Systems

Definition

A (Lagrangian) nonholonomic system, as studied by d'Alembert, appears naturally as a model for mechanical systems with rolling constraints and no dissipation of energy.

- Constraints on the velocity (rolling constraints)
- Represented by a distribution $C \subset TQ$

Example

A ball rolling on a table: the angular velocity and the translation velocity must be consistent with the condition that the point of contact has velocity 0.

Principle of Lagrange–d'Alembert

Motion of the system is a curve $q(t)$ such that

- $\dot{q} \in C$
- it is a critical point of $\int L\, dt$ with respect to $\delta q \in C$

Generalized Nonholonomic Systems (GNHS)

Extension of d'Alembert's Principle

Rolling systems with friction forces and dissipation of energy.
Optimal control problems (isoholonomic problem).

More General Version of the Principle of Virtual Work

Unfold the double role of C

1. Kinematic distribution C_K
2. Variational distribution C_V

Find a curve q such that

1. $\dot{q} \in C_K$
2. it is a critical point of $\int L\, dt$ with respect to variations $\delta q \in C_V$

(Cendra, Ibort, de León, Martín de Diego, J. Math. Phys. 2004) and references therein.

Lagrangian Systems

Definition

I will consider Lagrangian systems given by

- A manifold Q
- A Lagrangian $L\colon TQ \to \mathbb{R}$

A trajectory of it is a curve $q\colon [t_0, t_1] \to Q$ that is a critical point of the action

$$\int_{t_0}^{t_1} L(q, \dot{q})\, dt$$

with respect to variations δq with fixed endpoint conditions.

Deformations and variations

Definition

A family of curves $q_\lambda \colon [t_0, t_1] \to Q$, with λ in a neighborhood of 0 is called

- a deformation of the curve q if $q_0 = q$
- with fixed endpoints if each endpoint $q_\lambda(t_i)$, $i = 0, 1$, does not depend on λ

Definition

A variation $\delta q(t)$ of a curve $q(t)$ is a vector field along that curve

Deformations and variations

Induced variations

A deformation $q_\lambda(t)$ induces a variation

$$\delta q(t) = \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=0} q_\lambda(t)$$

Spaces of curves

- In many cases it is possible to work with C^∞ or piecewise-C^∞ curves and deformations.
- For the isoholonomic problem it is convenient to work with absolutely continuous curves (Sobolev space H^1.)

Principal bundles and holonomy

Definition

A principal bundle Q over X with structure group G is a manifold such that

- G acts freely on Q
- $X = Q/G$, canonical projection is a submersion $\pi \colon Q \to Q/G$
- Q is locally trivial; it is locally $U \times G$ where $U \subseteq X$ is open.

Cases where the action of the group is not free and the quotient X is singular are important and are the purpose of recent research.

Principal bundles and holonomy

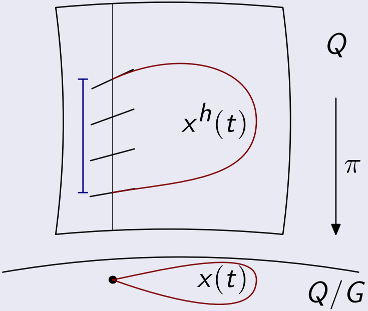
Definition

Principal connection $\mathcal{A}\colon TQ \to \mathfrak{g}$

- Equivariant: $\mathcal{A}(gv_q) = \mathrm{Ad}_g(\mathcal{A}(v_q))$
- $\mathcal{A}(\xi q) = \xi$, where $\xi \in \mathfrak{g}$

Holonomy

Graphical representation of a principal bundle



A curve in the base gives a holonomy.

- The curve must be at least absolutely continuous to have a well-defined horizontal lift.

Associated bundles

Definition

Let V be a representation of G. The associated bundle $\widetilde{V}$ is defined by $\widetilde{V} = (Q \times V)/G$.

Example

The associated bundle for the Lie algebra $\mathfrak{g}$ with the adjoint action $\mathrm{Ad}\colon G \times \mathfrak{g} \longrightarrow \mathfrak{g}$ is called the *adjoint bundle* $\widetilde{\mathfrak{g}}$.

Notation

$\bar{v} \equiv [q, v]_G$ (equivalence class of (q, v))

Associated bundles

Induced operations

Invariant or equivariant operations among representations pass to the quotient in a natural way.

- Pairing between V and V^*
- Lie bracket: $[\bar{\xi}, \bar{\eta}] = [q, [\xi, \eta]]_G$
- Infinitesimal generators ($\mathfrak{g} \times V \to V$)

Covariant derivative

The principal connection $\mathcal{A}$ in Q induces a covariant derivative of curves $\bar{v}(t)$ on $\widetilde{V}$:

$$\frac{D\bar{v}}{Dt} = [q, \dot{v} - \mathcal{A}(q, \dot{q})v]_G$$

Lagrangian systems with symmetry

Symmetry

- Lagrangian system on a principal bundle Q with structure group G
- G-invariant Lagrangian
- Some other structures (for instance, G-invariant distributions)

Reduction process

- Quotient space $(TQ)/G$
  - $L\colon TQ \to \mathbb{R} \quad \rightsquigarrow \quad [L]_G\colon (TQ)/G \to \mathbb{R}$
- Pass variational principle to the quotient

Advantages of reduction

- Simplifies equations of motion
- Reveals conservation laws

Lagrangian reduction

Quotient bundle and reduced Lagrangian

$$\begin{tikzcd}
TQ \arrow[r, two heads] \arrow[d, "L"'] & (TQ)/G \arrow[r, "\alpha_{\mathcal{A}}"] \arrow[dl, "{[L]_G}"'] & TX \oplus \tilde{\mathfrak{g}} \arrow[dll, "\ell"] \\
\mathbb{R} & &
\end{tikzcd}$$

Isomorphism $\alpha_{\mathcal{A}} \colon (TQ)/G \to TX \oplus \tilde{\mathfrak{g}}$

$$\alpha_{\mathcal{A}}([q, \dot{q}]_G) = T\pi(q, \dot{q}) \oplus [q, \mathcal{A}(q, \dot{q})]_G$$

Reduced Lagrangian $\ell = [L]_G \circ \alpha_{\mathcal{A}}^{-1}$

Reduced variations

Question

Given a deformation q_λ of q with fixed endpoints.
How to calculate the infinitesimal deformation of the corresponding quotient curve?

Answer

Define $(x, \dot{x}) \oplus \bar{v} = \alpha_{\mathcal{A}}([q, \dot{q}]_G) \in TX \oplus \widetilde{\mathfrak{g}}$.
We obtain

- $\delta x(t_i) = 0,\ i = 0, 1$
- $\delta^{\mathcal{A}} \bar{v} = \widetilde{B}(x)(\delta x, \dot{x})$ for δq horizontal
- $\delta^{\mathcal{A}} \bar{v} = \frac{D[q,\eta]_G}{Dt} + [q, [v, \eta]]_G$ for $\delta q = \eta q$ vertical, where η is a curve in $\mathfrak{g}$ which is 0 at the endpoints

Remark

For absolutely continuous curves some extra terms may appear which are 0 in L^2, so they do not affect the calculations.

Nonholonomic constraints

Nonholonomic constraint

- Choose a G-invariant metric on Q.
- Let $\mathcal{D}$ be a given invariant distribution on Q (nonholonomic constraint).
- ***Dimension assumption***

$$TQ = \mathcal{D} + \mathcal{V},$$

  where $\mathcal{V}$ is the vertical distribution.
- Let $\mathcal{S} = \mathcal{D} \cap \mathcal{V}$.

Principal connection

- Principal connection form $\mathcal{A} : TQ \to \mathfrak{g}$
- horizontal distribution $\mathrm{Hor}^{\mathcal{A}}\, TQ$
- the space $\mathrm{Hor}^{\mathcal{A}}\, T_qQ$ is the orthogonal complement $\mathcal{H}_q$ of the space $\mathcal{S}_q$ in $\mathcal{D}_q$.

Three distributions

Nonholonomic connection

This connection is called the ***nonholonomic connection***.

$\mathcal{U}_q$ orthogonal complement of $\mathcal{S}_q$ in $\mathcal{V}_q$.

- $TQ = \mathcal{H} \oplus \mathcal{S} \oplus \mathcal{U}$.
- $\mathcal{D} = \mathcal{H} \oplus \mathcal{S}$
- $\mathcal{V} = \mathcal{S} \oplus \mathcal{U}$.

All three distributions $\mathcal{H}$, $\mathcal{S}$, and $\mathcal{U}$ are G-invariant, so we can write,

$$TQ/G = \mathcal{H}/G \oplus \mathcal{S}/G \oplus \mathcal{U}/G.$$

Reduced bundle

It is easy to see that

- $\alpha_{\mathcal{A}}(\mathcal{H}/G) = T(Q/G)$,
- $\alpha_{\mathcal{A}}(\mathcal{V}/G) = \tilde{\mathfrak{g}}$.

Define the subbundles $\tilde{\mathfrak{s}}$ and $\tilde{\mathfrak{u}}$ of $\tilde{\mathfrak{g}}$ by

- $\tilde{\mathfrak{s}} = \alpha_{\mathcal{A}}(\mathcal{S}/G)$
- $\tilde{\mathfrak{u}} = \alpha_{\mathcal{A}}(\mathcal{U}/G)$

Clearly, we have

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{s}} \oplus \tilde{\mathfrak{u}}.$$

Main theorem on reduction of nonholonomic systems

Main theorem

Let $q(t)$ be a curve in Q such that $(q(t), \dot{q}(t)) \in \mathcal{D}_{q(t)}$ for all t and let $(x(t), \dot{x}(t), \bar{v}(t)) = \alpha_{\mathcal{A}}([q(t), \dot{q}(t)]_G)$ be the corresponding curve in $T(Q/G) \oplus \tilde{\mathfrak{s}}$. The following conditions are equivalent.

Equivalent conditions

(i) d'Alembert's principle holds

$$\delta \int_{t_0}^{t_1} L(q, \dot{q}) dt = 0$$

for variations δq of the curve q such that $\delta q(t_i) = 0$, for $i = 0, 1$, and $\delta q(t) \in \mathcal{D}_{q(t)}$, for all t.

(ii) Reduced d'Alembert's principle holds

The curve $x(t) \oplus \bar{v}(t)$ satisfies $\delta \int_{t_0}^{t_1} \ell \left(x(t), \dot{x}(t), \bar{v}(t)\right) dt = 0$, for variations $\delta x \oplus \delta^{\mathcal{A}} \bar{v}$ of the curve $x(t) \oplus \bar{v}(t)$, where $\delta^{\mathcal{A}} \bar{v}$ is

$$\delta^{\mathcal{A}} \bar{v} = \frac{D\bar{\eta}}{Dt} + [\bar{v}, \bar{\eta}] + \tilde{\mathcal{B}}(\delta x, \dot{x}),$$

$\delta x(t_i) =, 0 \ \bar{\eta}(t_i) = 0$, for $i = 0, 1$, and $\bar{\eta}(t) \in \tilde{\mathfrak{s}}_{x(t)}$.

More equivalent conditions

(iii) Lagrange-d'Alembert-Poincaré equations hold

Vertical variations:

$$\frac{D}{Dt}\frac{\partial \ell}{\partial \bar{v}}(x,\dot{x},\bar{v})\bigg|_{\tilde{\mathfrak{s}}} = \operatorname{ad}^*_{\bar{v}}\frac{\partial \ell}{\partial \bar{v}}(x,\dot{x},\bar{v})\bigg|_{\tilde{\mathfrak{s}}}$$

Horizontal variations:

$$\frac{\partial^C \ell}{\partial x}(x,\dot{x},\bar{v}) - \frac{D}{Dt}\frac{\partial \ell}{\partial \dot{x}}(x,\dot{x},\bar{v}) = \left\langle \frac{\partial \ell}{\partial \bar{v}}(x,\dot{x},\bar{v}), \mathbf{i}_{\dot{x}}\tilde{\mathcal{B}}(x) \right\rangle$$

Examples of d'Alembert type

Lagrange-d'Alembert-Poincaré equations

- In principle, they are adapted to apply numerical integrators (to be done).
- In examples, they lead to nice solutions.

Applying Lagrange-d'Alembert-Poincaré

- For the Euler disk: we obtain solutions in terms of hypergeometric functions (in collaboration with Viviana Díaz, accepted in RCD)
- Symmetric "elastic" sphere with Veselova's constraints: This example has some singularities. Applying techniques from DAE and desingularization we reduce the system to an ODE on $S^2 \times S^1$. Integrability in terms of Liouvillean functions obtained. (In collaboration with Marila Etchechoury, published recently in Reports on Math. Phys.)

Examples of Generalized-d'Alembert type

As said before, by allowing for constraints on the variations C_V to be different from constraints on the velocities C_K one can represent geometrically several systems which cannot be treated using d'Alembert's Principle.

Some examples

- Rolling with friction: tires, viscoelastic balls; in collaboration with A. Ibort, D. Martín de Diego and M. de León, J. Math. Phys. 2004.
- Strategy for optimal control of inverted pendulum; in collaboration with Sergio Grillo, J. Math. Phys, 2006.
- Isoholonomic problems, phases in mechanics; in collaboration with Sebastián Ferraro, Dyn. Systems: An International Journal, 2006.

In the next section I am going to describe our work with Sebastián Ferraro, as an example of generalized nonholonomic system. The isoholonomic problem is converted into a nonholonomic system of generalized-d'Alembert's type. This kind of formulation is adapted to reduction. One simply reduces each constraint, C_K and C_V, in a similar way as one does with d'Alembert's principle, where $C_K = C_V$. To deal with the isoholonomic problem one needs a generalization of the Lagrange-d'Alembert-Poincaré equations, to allow for parameter-dependent Lagrangians.

Lagrangian systems depending on a parameter

Idea

- Given Lagrangian system
- Some fixed parameter is considered as a dynamical variable.
  - Example: Vector representing the acceleration of gravity for the heavy top.
- Can introduce new symmetry in the system with parameter.
- Then one can reduce

Aplications

- Heavy top
- Fluids
- Plasmas

Systems depending on a parameter

Sketch

- Parameter space V^* (dual of a representation of G)
- β will typically denote a parameter.
- Lagrangian $L\colon TQ \times V^* \longrightarrow \mathbb{R}$ such that the given Lagrangian TQ coincides with $L(q, \dot{q}, \beta)$ for certain $\beta_0 \in V^*$

Systems depending on a parameter

Lemma

The following are equivalent

1. Find critical curves $q(t)$ of the action

$$\int_{t_0}^{t_1} L(q, \dot{q}, \beta_0)\, dt$$

   for β_0 fixed, with respect to variations of q with fixed endpoints.

2. Find critical curves q en Q, β in V^*, α in V, with $\beta(t_0) = \beta_0$ of the action

$$\int_{t_0}^{t_1} \left(L(q, \dot{q}, \beta) + \langle \dot{\beta}, \alpha \rangle \right)\, dt$$

   with respect to variations q and β with fixed endpoints and arbitrary variations of α

Reduction of parameter dependent Lagrangians

Symmetry

$L\colon TQ \times V^* \to \mathbb{R}$ is G-invariant:

$$L(gq, g\dot{q}, g\beta) = L(q, \dot{q}, \beta)$$

Reduced bundle

$$TQ \times V^* \equiv TQ \oplus (Q \times V^*)$$

$$\left(TQ \oplus (Q \times V^*)\right)/G \equiv (TQ)/G \oplus (Q \times V^*)/G \equiv TX \oplus \widetilde{\mathfrak{g}} \oplus \widetilde{V}^*$$

Reduced Lagrangian

$\ell\colon TX \oplus \widetilde{\mathfrak{g}} \oplus \widetilde{V}^* \to \mathbb{R}$ naturally defined by

$$\ell\big((x, \dot{x}) \oplus \bar{v} \oplus \bar{\beta}\big) = L\big((x, \dot{x})^h_q + vq, \beta\big)$$

Reduction of parameter dependent systems

Extended Lagrangians

- Find critical points of

$$\int_{t_0}^{t_1} \underbrace{\left(L(q,\dot{q},\beta) + \langle \dot{\beta}, \alpha \rangle\right)}_{\bar{L}} \, dt$$

- If L is G-invariant then $\bar{L}$ is also invariant.
- Reduced version of $\bar{L}$ is

$$\bar{\ell}\colon TX \oplus \tilde{\mathfrak{g}} \oplus \widetilde{V}^* \oplus \widetilde{V}^* \oplus \widetilde{V} \to \mathbb{R}$$

  and it has the expression

$$\ell\left((x,\dot{x}) \oplus \bar{v} \oplus \bar{\beta}\right) + \left\langle \frac{D\bar{\beta}}{Dt} + \bar{v}\bar{\beta}, \bar{\alpha} \right\rangle$$

Reduction of parameter dependent systems

Lemma

The following are equivalent

1. Find critical points of $\int L(q, \dot{q}, \beta_0)\, dt$ for fixed β_0
2. Find critical points of $\int \left(L(q, \dot{q}, \beta) + \langle \dot{\beta}, \alpha \rangle\right) dt$
3. Find critical points x, $\bar{v}$, $\bar{\beta}$, $\bar{\alpha}$, of

$$\int_{t_0}^{t_1} \left(\ell\left(x, \dot{x}, \bar{v}, \bar{\beta}\right) + \left\langle \frac{D\bar{\beta}}{Dt} + \bar{v}\bar{\beta}, \bar{\alpha} \right\rangle \right) dt$$

$$\delta x(t_i) = 0 \text{ for } i = 0, 1,$$

$$\delta^{\mathcal{A}} \bar{v} = \widetilde{B}(x)(\delta x, \dot{x}) + \frac{D\bar{\eta}}{Dt} + [\bar{v}, \bar{\eta}]\,,$$

$$\delta^{\mathcal{A}} \bar{\beta}(t_i) = 0 \text{ for } i = 0, 1,$$

where $\bar{\eta}$ is a curve in $\widetilde{\mathfrak{g}}$ vanishing at the endpoints

Reduction of parameter dependent systems

LP with parameters

$$\frac{D\bar{\beta}}{Dt} + \bar{v}\bar{\beta} = 0$$

$$-\frac{D}{Dt}\frac{\partial \ell}{\partial \bar{v}} + \mathrm{ad}^*_{\bar{v}}\frac{\partial \ell}{\partial \bar{v}} + \frac{\partial \ell}{\partial \bar{\beta}} \diamond \bar{\beta} = 0$$

$$\frac{\partial \ell}{\partial x} - \frac{D}{Dt}\frac{\partial \ell}{\partial \dot{x}} - \frac{\partial \ell}{\partial \bar{v}} \cdot \widetilde{B}(x)(\dot{x}, \cdot) = 0$$

Reduction of parameter dependent systems

LP with parameters

$$\frac{D\bar{\beta}}{Dt} + \bar{v}\bar{\beta} = 0$$

$$-\frac{D}{Dt}\frac{\partial \ell}{\partial \bar{v}} + \operatorname{ad}^*_{\bar{v}} \frac{\partial \ell}{\partial \bar{v}} + \frac{\partial \ell}{\partial \bar{\beta}} \diamond \bar{\beta} = 0$$

$$\frac{\partial \ell}{\partial x} - \frac{D}{Dt}\frac{\partial \ell}{\partial \dot{x}} - \frac{\partial \ell}{\partial \bar{v}} \cdot \widetilde{B}(x)(\dot{x}, \cdot) = 0$$

- Diamond $\diamond$: $V \times V^* \to \mathfrak{g}^*$
  $(\alpha \diamond \beta)(\xi) = \langle \beta, \xi\alpha \rangle$ for $\alpha \in V$, $\beta \in V^*$ y $\xi \in \mathfrak{g}$.
- Curvature 2-form
  $B(q)(u_q, v_q) = d\mathcal{A}\big(\mathrm{Hor}^{\mathcal{A}}(u_q), \mathrm{Hor}^{\mathcal{A}}(v_q)\big) \in \mathfrak{g}$
- $\widetilde{B}(x)(\dot{x}, \delta x) = [q, B(q)(\dot{q}, \delta q)]_G \in \widetilde{\mathfrak{g}}$

Reduction of parameter dependent systems

LP with parameters

$$\frac{D\bar{\beta}}{Dt} + \bar{v}\bar{\beta} = 0$$

$$-\frac{D}{Dt}\frac{\partial \ell}{\partial \bar{v}} + \operatorname{ad}^*_{\bar{v}} \frac{\partial \ell}{\partial \bar{v}} + \frac{\partial \ell}{\partial \bar{\beta}} \diamond \bar{\beta} = 0$$

$$\frac{\partial \ell}{\partial x} - \frac{D}{Dt}\frac{\partial \ell}{\partial \dot{x}} - \frac{\partial \ell}{\partial \bar{v}} \cdot \widetilde{B}(x)(\dot{x}, \cdot) = 0$$

No parameters

If the parameter space is 0, some terms vanish and Lagrange–Poincaré equations (Cendra, Marsden, Rațiu, Memoirs AMS, 2001) are recovered.

Reduction of parameter dependent systems

LP with parameters

$$\frac{D\bar{\beta}}{Dt} + \bar{v}\bar{\beta} = 0$$

$$-\frac{D}{Dt}\frac{\partial \ell}{\partial \bar{v}} + \mathrm{ad}^*_{\bar{v}}\frac{\partial \ell}{\partial \bar{v}} + \frac{\partial \ell}{\partial \bar{\beta}} \diamond \bar{\beta} = 0$$

$$\frac{\partial \ell}{\partial x} - \frac{D}{Dt}\frac{\partial \ell}{\partial \dot{x}} - \frac{\partial \ell}{\partial \bar{v}} \cdot \widetilde{B}(x)(\dot{x}, \cdot) = 0$$

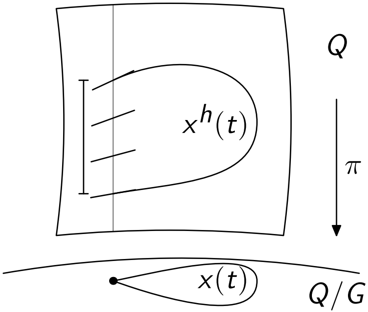
These are some points in Ferraro's thesis

- Generalize Lagrange–Poincaré
- Generalize earlier works for trivial bundles
- Covariant formulation

Isoholonomic problems

Description

- Principal bundle Q, structure group G
- Principal connection $\mathcal{A}\colon TQ \to \mathfrak{g}$
- Riemannian metric k in the base manifold $X = Q/G$



Statement of the problem

Find a curve $x\colon [t_0, t_1] \to X$
minimizing $\int \sqrt{k(\dot{x}, \dot{x})}\, dt$
in the family of curves
with given holonomy.

Can give an equivalent statement in terms of horizontal curves in Q between two given points.

Isoholonomic problems

Family of curves

For a given curve the isoholonomic problem requires to consider, for a given curve, deformations with fixed endpoints and fixed holonomy.

Are there some such deformations?

- For C^1 curves this is not always the case (*rigid curves*, Bryant and Hsu, Invent. Math. 1993)
- For absolutely continuous curves, yes, (if the horizontal distribution is bracket generating)

Advantage of working with absolutely continuous curves

- It is a class of curves with enough deformations
- Sufficiently regular to ensure basic properties, like equality of cross derivatives in L^2

The bi-invariant case

Theorem (Montgomery, Comm. Math. Phys. 1990)

If G admits a bi-invariant metric β, define a metric $k \oplus \beta$ on Q,

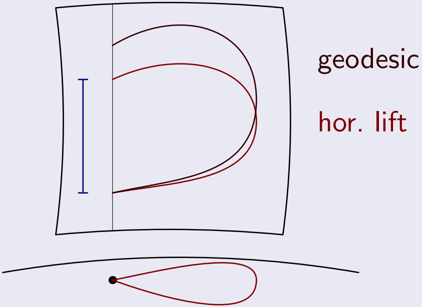
$$(k \oplus \beta)(u_q, v_q) = k(T\pi(u_q), T\pi(v_q)) + \beta\left(\mathcal{A}(u_q), \mathcal{A}(v_q)\right).$$

Then solutions to the isoholonomic problem are the projections of the geodesics in Q.

Geodesics

Observation

Geodesics are *not* horizontal.



Arbitrary groups

Groups with no bi-invariant metrics.

For example Euclidean groups SE(2), SE(3)

Generalization

- Generalization of the theorem for arbitrary groups.
- Class of (GNHS) in the context of parameter dependent Lagrangians
- Solutions to the isoholonomic problem are projections of solutions to the GNHS
- If there is a bi-invariant metric those solutions are Montgomery's geodesics.

Parameter space

Definition

M space of all symmetric bilinear forms on $\mathfrak{g}$. For $g \in G$, $\beta \in M$ and $\eta_1, \eta_2 \in \mathfrak{g}$ define

$$(g\beta)(\eta_1, \eta_2) = \beta\left(\mathrm{Ad}_{g^{-1}}\eta_1, \mathrm{Ad}_{g^{-1}}\eta_2\right)$$

Infinitesimal generator

If $\xi \in \mathfrak{g}$ and $\beta \in M$, then the infinitesimal generator at β is

$$(\xi\beta)(\eta_1, \eta_2) = \beta(-[\xi, \eta_1], \eta_2) + \beta(\eta_1, -[\xi, \eta_2])$$

If β is bi-invariant then $\xi\beta = 0$ for all ξ

- Define the associated bundle $\widetilde{M} = (Q \times M)/G$
- We write $\bar{\beta} = [q, \beta]_G \in \widetilde{M}$

First approach to the problem

Lagrange multipliers

- Find critical points of $\int_{t_0}^{t_1} \sqrt{k\left(T\pi(\dot{q}), T\pi(\dot{q})\right)}\, dt$ under horizontality condition $\mathcal{A}(q, \dot{q}) = 0$
- Fix a metric β and define

$$S\,(q, e) = \int_{t_0}^{t_1} \sqrt{k\left(T\pi(\dot{q}), T\pi(\dot{q})\right)}\, dt + \int_{t_0}^{t_1} \beta(e, \mathcal{A}(q, \dot{q}))\, dt,$$

where $e(t)$ is a curve in $\mathfrak{g}$.
We obtain

$$\frac{D\bar{\beta}}{Dt} = 0$$
$$\frac{D\bar{e}}{Dt} = 0$$
$$(\nabla_{\dot{x}}\dot{x})^{\flat} = -\bar{\beta}(\bar{e}, \widetilde{B}(x)(\dot{x}, \cdot))$$

Nonholonomic systems

Definition

Nonholonomic (Lagrangian) system

- Restriction on velocities (rolling restrictions)
- Represented by a distribution $C \subset TQ$

Principle of Lagrange–d'Alembert

Find curve $q(t)$ such that

- $\dot{q} \in C$
- is a critical point of $\int L\,dt$ with respect to $\delta q \in C$

Double role of C

Generalized nonholonomic systems (GNHS)

Generalization

1. Kinematic distribution C_K
2. Variational distribution C_V

Find curve such that

1. $\dot{q} \in C_K$
2. be a critical point of $\int L\,dt$ with respect to variations $\delta q \in C_V$

(Cendra, Ibort, de León, Martín de Diego, J. Math. Phys. 2004)

The problem as a generalized nonholonomic system

The system

- Is a lagrangian system on the manifold $Q \times M$
- $L\colon\ T(Q \times M) \to \mathbb{R}$,

$$L(q, \beta, \dot{q}, \dot{\beta}) = \frac{1}{2} k(\dot{x}, \dot{x}) + \frac{1}{2} \beta \left(\mathcal{A}(q, \dot{q}), \mathcal{A}(q, \dot{q}) \right)$$

- Distributions of $T(Q \times M)$
  - Kinematic $C_K :\ \ \dot{\beta} = \mathcal{A}(q, \dot{q})\beta$
  - Variational $C_V :\ \ \delta\beta = 0$
- The Lagrangian and the distributions are G-invariant

Reduction

Reduced bundle and Lagrangian

$$T(Q \times M) \equiv TQ \times TM \equiv TQ \oplus (Q \times M) \oplus (Q \times M)$$

$$\big(T(Q \times M)\big)/G \equiv TX \oplus \tilde{\mathfrak{g}} \oplus 2\widetilde{M}$$

The reduced Lagrangian $\ell\colon TX \oplus \tilde{\mathfrak{g}} \oplus 2\widetilde{M} \rightarrow \mathbb{R}$ is

$$\ell\left((x, \dot{x}) \oplus \bar{v} \oplus \bar{\beta} \oplus \bar{\beta}'\right) = \frac{1}{2}k(\dot{x}, \dot{x}) + \frac{1}{2}\bar{\beta}(\bar{v}, \bar{v})$$

Reduced distributions

- Kinematic reduced distribution: $\dfrac{D\bar{\beta}}{Dt} = 0$
- Variational reduced distribution: $\delta^{\mathcal{A}}\bar{\beta} = -\bar{\eta}\bar{\beta}$

Reduction

Reduced variations

$$\begin{aligned}\delta x &= T\pi(\delta q)\\ \delta^{\mathcal{A}}\bar{v} &= \widetilde{B}(x)(\delta x, \dot{x}) + \frac{D\bar{\eta}}{Dt} + [\bar{v}, \bar{\eta}]\\ \delta^{\mathcal{A}}\bar{\beta} &= -\bar{\eta}\bar{\beta}\end{aligned}$$

Reduced equations for the GNHS

$$\begin{aligned}\frac{D\bar{\beta}}{Dt} &= 0\\ \frac{D\bar{v}}{Dt} &= 0\\ (\nabla_{\dot{x}}\dot{x})^{\flat} &= -\bar{\beta}\big(\bar{v}, \widetilde{B}(x)(\dot{x}, \cdot)\big)\end{aligned}$$

Changing $\bar{v}$ by $\bar{e}$, these equations coincide with those obtained with Lagrange multipliers

Comparison of results

Lagrange multipliers

Solutions are horizontal curves and $\bar{e}(t)$ simply represents a multiplier

GNHS

Solutions are not horizontal and $\bar{v}(t)$ represents the vertical component

Comparison of results

- Reduced equations are the same but solutions should be interpreted differently.
- The *non* reduced problems are not directly equivalent.
- They are related through the reduction

The bi-invariant case

The geodesic theorem as a particular case

- If we take β_0 bi-invariant as initial value for $\beta(t)$, then from the kinematic distribution one obtains

$$\dot{\beta} = \mathcal{A}(q,\dot{q})\beta = 0$$

- Then we are finding the critical points of

$$\int_{t_0}^{t_1} \left(\frac{1}{2} k\left(\dot{x},\dot{x}\right) + \frac{1}{2}\beta_0\left(\mathcal{A}(q,\dot{q}),\mathcal{A}(q,\dot{q})\right)\right) dt$$

  for variations of q with fixed endpoints
- The solution is a geodesic for the Riemannian metric $k \oplus \beta_0$ of Montgomery's theorem

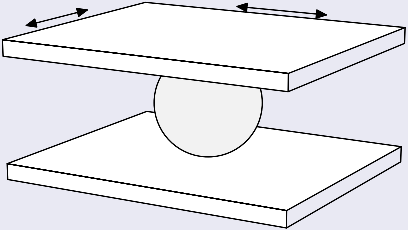
The generalized nonholonomic approach

We obtain

- Generalization of a theorem of Montgomery
- Can take advantage of the variational structure to obtain an algorithm to solve the isoholonomic problems of optimal control and nonholonomic systems

Optimal control of the plate-ball system

Definition of the system



The massless sphere rolls

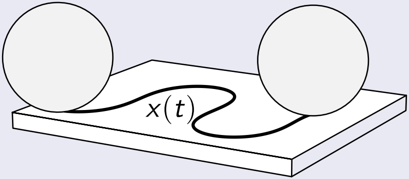
- without sliding
- without spinning (Veselova's constraint)

It is called *plate-ball system* in the control literature

- Robotic manipulation (*dextrous manipulation*)

Optimal control of the plate-ball system

Definition of the system



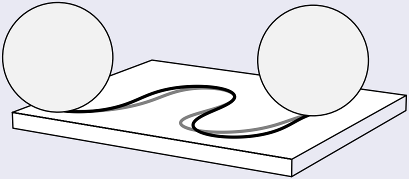
- The ball rolls along $x(t)$, $t \in [t_0, t_1]$
- A reorientation is produced depending on the trajectory.
- Several trajectories may give rise to the same reorientation.

Optimal control problem

Among all trajectories joining two given points and giving rise to the same reorientation find the one of minimal length

Optimal control of the plate-ball system

Definition of the system



- The ball rolls along $x(t)$, $t \in [t_0, t_1]$
- A reorientation is produced depending on the trajectory.
- Several trajectories may give rise to the same reorientation.

Optimal control problem

Among all trajectories joining two given points and giving rise to the same reorientation find the one of minimal length

Optimal control of the plate-ball system

Modelling the system

- Configuration space $Q = \mathrm{SO}(3) \times \mathbb{R}^2$
- $(A(t), x(t))$ evolution of the system
- Rolling constraint $\omega = \mathbf{e}_3 \times \dot{x}$ (spatial angular velocity)

Identification $\mathbb{R}^3 \equiv \mathfrak{so}(3)$

$$v \mapsto \hat{v} = \begin{bmatrix} 0 & -v^3 & v^2 \\ v^3 & 0 & -v^1 \\ -v^2 & v^1 & 0 \end{bmatrix}$$

$$\hat{\omega} = \dot{A}A^{-1} \in \mathfrak{so}(3)$$

Optimal control of the plate-ball system

Geometrization

- $Q = \mathrm{SO}(3) \times \mathbb{R}^2$ principal bundle
- $G = \mathrm{SO}(3)$ acts by $g(A, x) = (Ag^{-1}, x)$
- $X = Q/G \equiv \mathbb{R}^2$ base of the bundle
- Rolling restriction $\omega = \mathbf{e}_3 \times \dot{x} \longrightarrow$ distribution on TQ
  - G-invariant
  - complements the vertical distribution
- Principal connection

$$\mathcal{A}(A, x, \dot{A}, \dot{x}) = -A^{-1}\dot{A} + A^{-1}(\widehat{\mathbf{e}_3 \times \dot{x}})A$$

Infinitesimal motion allowed $\mathcal{A} = 0$ (horizontal)

The optimal control problem is an isoholonomic problem

Optimal control of the plate-ball system

Solution

- M = Space of symmetric bilinear forms in $\mathfrak{so}(3) \equiv \mathbb{R}^3$ (symmetric matrices 3×3)
- Equations

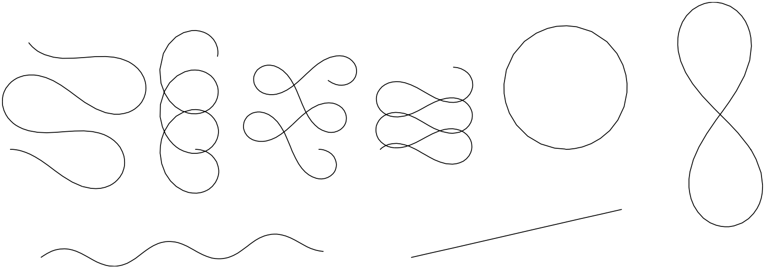
$$\begin{aligned} \ddot{x} &= (v \cdot \beta_{.3})\, \mathbf{e}_3 \times \dot{x} \\ \dot{v} &= (\mathbf{e}_3 \times \dot{x}) \times v \\ \dot{\beta} &= \left[\widehat{\mathbf{e}_3 \times \dot{x}}, \beta\right] \end{aligned}$$

- For $\beta = \mathrm{Id}_{3\times 3}$:

$$\begin{aligned} \dot{v} &= (\mathbf{e}_3 \times \dot{x}) \times v \\ \ddot{x} &= v^3 \mathbf{e}_3 \times \dot{x} \end{aligned}$$

Optimal control of the plate-ball system

Some typical solutions



Optimal control of the plate-ball system

Analogy with the pendulum dynamics

$$\ddot{x} = v^3 \mathbf{e}_3 \times \dot{x} \Rightarrow \|\dot{x}\| = \text{constant}$$

$$\dot{x}(t) = R\left(\cos\varphi(t), \sin\varphi(t), 0\right)$$

Substitution in the equations gives

$$\ddot{\varphi} = -\omega_0^2 \sin\left(\varphi - \alpha\right)$$

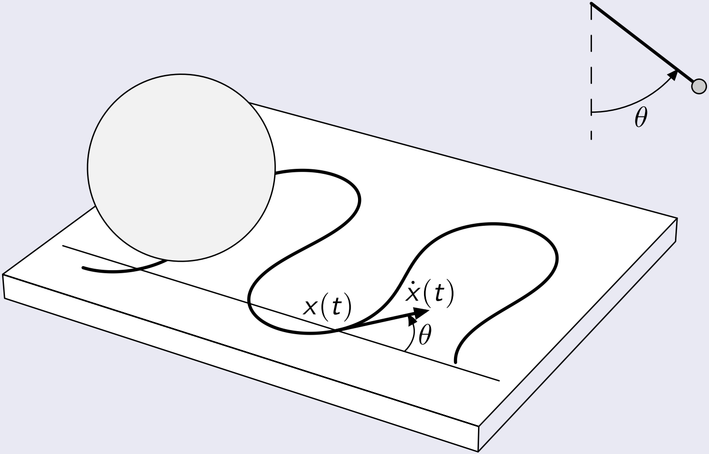
Writing $\theta = \varphi - \alpha$, we obtain

$$\ddot{\theta} = -\omega_0^2 \sin\theta$$

Equation of the pendulum, $\omega_0^2 = g/L$.

Optimal control of the plate-ball system

Analogy with pendulum dynamics



Microswimming (locomotion of microorganisms)

- High viscosity, inertia assumed to be zero.
- Q is the space of embeddings of S^2 into $\mathbb{R}^3$
- $G = \mathrm{SE}(3)$ (translations and rotations)
- An allowed infinitesimal motion must have zero force and torque on the fluid (Stokes connection)
- A given sequence of deformations of the membrane gives rise to a translation and a rotation (holonomy)
- The metric in Q/G measures the energy expenditure

Optimal microswimming

- Find a sequence of deformations giving rise to a given translation and rotation, with minimum cost of energy.
- Our approach can be applied to the problem (including discretization,...)