# Variational Calculus on Lie Algebroids

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# Abstract

We will prove that Lagrange's equations for a Lagrangian system on a Lie algebroid are precisely the equations for the critical points of the action functional defined on the set of admissible curves on a Lie algebroid with fixed base endpoints.

# Lie Algebroids

A Lie algebroid structure on the vector bundle $\tau\colon E \to M$ is given by

- a Lie algebra structure $(\mathrm{Sec}(E), [\,,\,])$ on the set of sections of $E$, and
- a morphism of vector bundles $\rho\colon E \to TM$ over the identity, such that
  - $\rho([\sigma, \eta]) = [\rho(\sigma), \rho(\eta)]$
  - $[\sigma, f\eta] = f[\sigma, \eta] + (\rho(\sigma)f)\,\eta$,

  where $\rho(\sigma)(m) = \rho(\sigma(m))$.

The first condition is actually a consequence of the second and the Jacobi identity.

# Examples

- **Tangent bundle.**

  $E = TM$,

  $\rho = \mathrm{id}$,

  $[\ ,\ ] =$ bracket of vector fields.

- **Integrable subbundle.**

  $E \subset TM$, integrable distribution

  $\rho = i$, canonical inclusion

  $[\ ,\ ] =$ restriction of the bracket to vector fields in $E$.

## Lie algebra.

$E = \mathfrak{g} \to M = \{e\}$, Lie algebra (fiber bundle over a point)
$\rho = 0$, trivial map (since $TM = \{0_e\}$)
$[\, , \,] =$ the bracket in the Lie algebra.

## Atiyah algebroid.

Let $\pi \colon Q \to M$ a principal $G$-bundle.
$E = TQ/G \to M$, (Sections are equivariant vector fields)
$\rho([v]) = T\pi(v)$ induced projection map
$[\, , \,] =$ bracket of equivariant vectorfields (is equivariant).

## Transformation Lie algebroid.

Let $\Phi\colon \mathfrak{g} \to \mathfrak{X}(M)$ be an action of a Lie algebra $\mathfrak{g}$ on $M$.

$E = M \times \mathfrak{g} \to M$,

$\rho(m, \xi) = \Phi(\xi)(m)$ value of the fundamental vectorfield

$[\,,\,] =$ induced by the bracket on $\mathfrak{g}$.

# Mechanics on Lie algebroids

(Weinstein 1996, Martínez 2001, de León *et. al.* 2004)

Lie algebroid $E \to M$.

$L \in C^\infty(E)$ or $H \in C^\infty(E^*)$

- $E = TM \to M$ Standard classical Mechanics
- $E = \mathfrak{D} \subset TM \to M$ (integrable) System with holonomic constraints
- $E = TQ/G \to M = Q/G$ System with symmetry
- $E = \mathfrak{g} \to \{e\}$ System on a Lie algebra
- $E = M \times \mathfrak{g} \to M$ System on a semidirect product (ej. heavy top)

# Structure functions

A local coordinate system $(x^i)$ in the base manifold $M$ and a local basis of sections $(e_\alpha)$ of $E$, determine a local coordinate system $(x^i, y^\alpha)$ on $E$.

The anchor and the bracket are locally determined by the local functions $\rho^i_\alpha(x)$ and $C^\alpha_{\beta\gamma}(x)$ on $M$ given by

$$\rho(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i}$$

$$[e_\alpha, e_\beta] = C^\gamma_{\alpha\beta}\; e_\gamma.$$

The function $\rho^i_\alpha$ and $C^\alpha_{\beta\gamma}$ satisfy some relations due to the compatibility condition and the Jacobi identity which are called the structure equations:

$$\rho^j_\alpha \frac{\partial \rho^i_\beta}{\partial x^j} - \rho^j_\beta \frac{\partial \rho^i_\alpha}{\partial x^j} = \rho^i_\gamma C^\gamma_{\alpha\beta}$$

$$\sum_{\mathrm{cyclic}(\alpha,\beta,\gamma)} \left[ \rho^i_\alpha \frac{\partial C^\nu_{\beta\gamma}}{\partial x^i} + C^\mu_{\beta\gamma} C^\nu_{\alpha\mu} \right] = 0.$$

# Exterior differential

On 0-forms

$$df(\sigma) = \rho(\sigma)f$$

On $p$-forms ($p > 0$)

$$d\omega(\sigma_1, \ldots, \sigma_{p+1}) =$$
$$= \sum_{i=1}^{p+1} (-1)^{i+1} \rho(\sigma_i)\omega(\sigma_1, \ldots, \widehat{\sigma_i}, \ldots, \sigma_{p+1})$$
$$- \sum_{i<j} (-1)^{i+j} \omega([\sigma_i, \sigma_j], \sigma_1, \ldots, \widehat{\sigma_i}, \ldots, \widehat{\sigma_j}, \ldots, \sigma_{p+1}).$$

# Exterior differential-local

Locally determined by

$$dx^i = \rho^i_\alpha e^\alpha$$

and

$$de^\alpha = -\frac{1}{2} C^\alpha_{\beta\gamma} e^\beta \wedge e^\gamma .$$

The structure equations are

$$d^2 x^i = 0 \quad \text{and} \quad d^2 e^\alpha = 0.$$

# Admissible maps and Morphisms

A bundle map $\Phi$ between $E$ and $E'$ is said to be admissible map if

$$\Phi^\star df = d\Phi^\star f.$$

A bundle map $\Phi$ between $E$ and $E'$ is said to be a morphism of Lie algebroids if

$$\Phi^\star d\theta = d\Phi^\star \theta.$$

Obviously every morphism is an admissible map.

# Lagrange's equations

Given a function $L \in C^\infty(E)$, we define a dynamical system on $E$ by means of a system of differential equations, which in local coordinates reads

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) + \frac{\partial L}{\partial y^\gamma} C^\gamma_{\alpha\beta} y^\beta = \rho^i_\alpha \frac{\partial L}{\partial x^i}$$
$$\dot{x}^i = \rho^i_\alpha y^\alpha.$$

The equation $\dot{x}^i = \rho^i_\alpha y^\alpha$ is the local expression of the admissibility condition: A curve $a\colon \mathbb{R} \to E$ is said to be admissible if

$$\rho \circ a = \frac{d}{dt}(\tau \circ a).$$

In other words, if $a\, dt\colon T\mathbb{R} \to E$ is an admissible map.

# Prolongation

Given a Lie algebroid $\tau\colon E \to M$ we can construct the $E$-tangent to $E$ (the prolongation of $E$). It is the vector bundle $\tau_1\colon \mathcal{T}^E E \to E$ where the fibre over $a \in E$ is

$$\mathcal{T}^E_a E = \{\, (b, v) \in E_m \times T_a E \mid T\tau(v) = \rho(b) \,\}$$

where $m = \tau(a)$.

Redundant notation: $(a, b, v)$ for the element $(b, v) \in \mathcal{T}^E_a E$.

The bundle $\mathcal{T}^E E$ can be endowed with a structure of Lie algebroid. The anchor $\rho^1 \colon \mathcal{T}^E E \to TE$ is just the projection onto the third factor $\rho^1(a, b, v) = v$.

The structure of Lie algebroid in $\mathcal{T}^E E$ can be defined in terms of the brackets of vertical and complete lifts

$$[\eta^C, \sigma^C] = [\sigma, \eta]^C, \qquad [\eta^C, \sigma^V] = [\sigma, \eta]^V \quad \text{and} \quad [\eta^V, \sigma^V] = 0.$$

# Geometric Lagrangian Mechanics

Associated to $L$ there is a section $\theta_L$ of $(\mathcal{T}^E E)^*$,

$$\langle \theta_L, \eta^C \rangle = d_{\eta^V} L \qquad \text{and} \qquad \langle \theta_L, \eta^V \rangle = 0.$$

A solution of Lagrange's equations is an admissible curve $a \colon \mathbb{R} \to E$ which satisfies $\delta L(\dot{a}(t)) = 0$, where

$$\langle \delta L(\dot{a}(t)), \eta(\gamma(t)) \rangle = \big(d_{\eta^C} L\big)(a(t)) - \frac{d}{dt}\big(\langle \theta_L, \eta^C \rangle (a(t))\big)$$

for every $\eta \in \mathrm{Sec}_\gamma(E)$ and where $\gamma = \tau \circ a$ is the curve on the base.

Variations $\longleftrightarrow$ Complete lifts

# The canonical involution

There exists a canonical involution $\chi_E\colon \mathcal{T}^E E \to \mathcal{T}^E E$.

It is defined by $\chi_E(a, b, v) = (b, a, \bar{v})$, for every $(a, b, v) \in \mathcal{T}^E E$, where $\bar{v} \in T_b E$ is the vector which projects to $\rho(a)$ and satisfies

$$\bar{v}\hat{\theta} = v\hat{\theta} + d\theta(a, b)$$

for every section $\theta$ of $E^*$.

In terms of the canonical involution, the complete lift of a section $\eta \in \mathrm{Sec}(E)$ is given by

$$\eta^C(a) = \chi_E\big(\eta(m), a, T_m\eta(\rho(a))\big).$$

# The map $Xi$

Given an admissible curve $a$ in $E$ over $\gamma = \tau \circ a$ we consider the map $\Xi_a \colon \operatorname{Sec}_\gamma(E) \to \operatorname{Sec}_a(TE)$ given by

$$\Xi_a(\sigma) = \rho^1(\chi_E(\sigma, a, \dot{\sigma})).$$

From the definition it is easy to prove the following property

$$\Xi_a(f\sigma) = f\Xi_a(\sigma) + \dot{f}\sigma_a^v,$$

for every function $f \in C^\infty(\mathbb{R})$.

Complete lifts can be obtained in terms of the above map. If $a(t)$ is an admissible curve over $\gamma$, then $\eta^C(a(t)) = \chi_E\big(\eta(\gamma(t)), a(t), \frac{d}{dt}(\eta(\gamma(t)))\big)$. In other words,

$$\rho^1(\eta^C) \circ a = \Xi_a(\eta \circ \gamma).$$

# Local expressions

Complete lift of $\eta = \eta^\alpha e_\alpha$

$$\rho^1(\eta^C) = \rho^i_\alpha \eta^\alpha + \left(\rho^i_\beta y^\beta \frac{\partial \eta^\alpha}{\partial x^i} + C^\alpha_{\beta\gamma} y^\beta \eta^\gamma\right) \frac{\partial}{\partial y^\alpha}$$

The canonical involution

$$\chi_E(x^i, y^\alpha, z^\alpha, v^\alpha) = (x^i, z^\alpha, y^\alpha, v^\alpha + C^\alpha_{\beta\gamma} z^\beta y^\gamma)$$

The map $\Xi_a$ is

$$\Xi_a(\sigma)(t) = \rho^i_\alpha(\gamma(t))\sigma^\alpha(t) \frac{\partial}{\partial x^i}\Big|_{(\gamma^i, a^\alpha)} + \\ + \left(\dot\sigma^\alpha(t) + C^\alpha_{\beta\gamma}(\gamma(t)) a^\beta(t) \sigma^\gamma(t)\right) \frac{\partial}{\partial y^\alpha}\Big|_{(\gamma^i, a^\alpha)}$$

where $a(t) = (\gamma^i(t), a^\alpha(t))$ and $\sigma(t) = (\gamma^i(t), \sigma^\alpha(t))$.

# $E$-Homotopy

Let $I = [0, 1]$ and $J = [t_0, t_1]$, and $(s, t)$ coordinates in $\mathbb{R}^2$.

**Definition 1** *Two $E$-paths $a_0$ and $a_1$ are said to be $E$-homotopic if there exists a morphism of Lie algebroids $\Phi \colon TI \times TJ \to E$ such that*

$$\Phi\left(\left.\frac{\partial}{\partial t}\right|_{(0,t)}\right) = a_0(t) \qquad \Phi\left(\left.\frac{\partial}{\partial s}\right|_{(s,t_0)}\right) = 0$$
$$\Phi\left(\left.\frac{\partial}{\partial t}\right|_{(1,t)}\right) = a_1(t) \qquad \Phi\left(\left.\frac{\partial}{\partial s}\right|_{(s,t_1)}\right) = 0.$$

It follows that the base map is a homotopy (in the usual sense) with fixed endpoints between the base paths.

# $E$-Homotopy.

Given a vector bundle map $\Phi\colon T\mathbb{R}^2 \to E$, denote $a(s,t) = \Phi(\partial_t|_{(s,t)})$ and $b(s,t) = \Phi(\partial_s|_{(s,t)})$, so that we can write $\Phi = adt + bds$.

An $E$-path $a_0$ is $E$-homotopic to an $E$-path $a_1$ if and only if there exists $a(s,t)$ and $b(s,t)$ such that

1. $a(0,-) = a_0$ and $a(1,-) = a_1$ (i.e. $a(s,t)$ is a homotopy from $a_0$ to $a_1$).
2. $b(s,t_0) = 0$ and $b(s,t_1) = 0$.
3. $t \mapsto a(s,t)$ and $s \mapsto b(s,t)$ are admissible curves.
4. $d\theta(a,b) = \frac{\partial}{\partial t}\langle\theta,b\rangle - \frac{\partial}{\partial s}\langle\theta,a\rangle$ for every section $\theta$ of $E^*$.

# $E$-Homotopy..

Equivalently:

An $E$-path $a_0$ is $E$-homotopic to an $E$-path $a_1$ if and only if there exists $a(s,t)$ and $b(s,t)$ such that

1. $a(0,-) = a_0$ and $a(1,-) = a_1$ (i.e. $a(s,t)$ is a homotopy from $a_0$ to $a_1$).
2. $b(s,t_0) = 0$ and $b(s,t_1) = 0$.
3. $t \mapsto a(s,t)$ and $s \mapsto b(s,t)$ are admissible curves.
4. $\chi_E\left(b, a, \frac{\partial b}{\partial t}\right) = \left(a, b, \frac{\partial a}{\partial s}\right)$.

# Construction of $E$-Homotopies

Let $a_0$ be an $E$-path, with base path $\gamma_0$, and $\eta \in \mathrm{Sec}(E)$. Let $(\Phi_s, \varphi_s)$ be the flow of the section $\eta$ and define

$$\gamma(s,t) = \varphi_s(\gamma_0(t))$$

$$a(s,t) = \Phi_s(a_0(t)) \qquad \text{and} \qquad b(s,t) = \eta(\gamma(s,t)).$$

Then $\xi = a(s,t)dt + b(s,t)ds$ is a morphism over $\gamma$.

Let $m_0, m_1 \in M$ and let $\eta \in \mathrm{Sec}(E)$ with compact support such that $\eta(m_0) = \eta(m_1) = 0$. Let $a_0$ be a curve such that its base curve connects the point $m_0$ with $m_1$. Then the map $\xi$ is an $E$-homotopy from $a_0$ to $a_1 = \Phi_1 \circ a_0$.

Can be extended for time-dependent sections.

# $E$-path space

- Consider the set of $E$-paths

$$\mathcal{A}(J,E) = \left\{ a\colon J \to E \;\middle|\; \rho \circ a = \frac{d}{dt}(\tau \circ a) \right\}.$$

The $E$-homotopy equivalence relation defines a partition of the space of $E$-paths into disjoint sets. We will see that every $E$-homotopy class is a smooth Banach manifold and that such partition is a foliation.

- $\mathcal{A}(J,E)$ is a Banach submanifold of the Banach manifold of $C^1$-paths whose base path is $C^2$.

- For $a \in \mathcal{A}(J,E)$, define the vector space

  $$\Sigma_\gamma = \left\{ \sigma \in \mathrm{Sec}_a(E) \;\middle|\; \sigma \text{ is } C^2 \text{ with } \sigma(t_0) = 0 \text{ and } \sigma(t_1) = 0 \right\},$$

  where $\gamma = \tau \circ a$. Define also the vector space $F_a \subset T_a\mathcal{A}(J,E)$ by $F_a = \Xi_a(\Sigma_\gamma)$ and finally $F = \cup_{a \in \mathcal{A}(J,E)} F_a \subset T\mathcal{A}(J,E)$.

- $F$ is a smooth integrable subbundle of the tangent bundle to $\mathcal{A}(J,E)$ and the leaves of the foliation defined by $F$ are precisely the $E$-homotopy classes. The codimension of $F$ is equal to $\dim(E)$.

- The $E$-path space with the appropriate differential structure is

  $$\boxed{\mathcal{P}(J,E) = \mathcal{A}(J,E)_F}.$$

# Global Frobenius theorem

**Theorem 1 (Lang)** *Let $F \subset TX$ be an integrable vector subbundle of $TX$. Using the restrictions of distinguished charts to plaques as charts we get a new structure of a smooth manifold on $X$, which we denote by $X_F$. If $F \neq TX$ the topology of $X_F$ is finer than that of $X$. $X_F$ has uncountably many connected components, which are the leaves of the foliation, and the identity induces an injective immersion $i\colon X_F \to X$.*

*If $f\colon Y \to X$ is a smooth map such that $Tf(TY) \subset F$, then the induced map $f_F\colon Y \to X_F$ (same values $f_F(x) = f(x)$ but different differentiable structure on the target space) is also smooth.*

# Variational description

Fix $m_0, m_1 \in M$ and consider the set of $E$-paths with such base endpoints

$$\mathcal{P}(J,E)_{m_0}^{m_1} = \{\, a \in \mathcal{P}(J,E) \mid \tau(a(t_0)) = m_0 \quad \text{and} \quad \tau(a(t_1)) = m_1 \,\}$$

It is a Banach submanifold of $\mathcal{P}(J,E)$.

**Theorem 2** *Let $L \in C^\infty(E)$ be a Lagrangian function on the Lie algebroid $E$ and fix two points $m_0, m_1 \in M$. Consider the action functional $S \colon \mathcal{P}(J,E) \to \mathbb{R}$ given by $S(a) = \int_{t_0}^{t_1} L(a(t))dt$. The critical points of $S$ on the Banach manifold $\mathcal{P}(J,E)_{m_0}^{m_1}$ are precisely those elements of that space which satisfy Lagrange's equations.*

# Morphisms and reduction

Given a morphism of Lie algebroids $\Phi\colon E \to E'$ the induced map $\hat{\Phi}\colon \mathcal{P}(J, E) \to \mathcal{P}(J, E')$ given by $\hat{\Phi}(a) = \Phi \circ a$ is smooth.

- If $\Phi$ is fiberwise surjective then $\hat{\Phi}$ is a submersion.
- If $\Phi$ is fiberwise injective then $\hat{\Phi}$ is a immersion.

Consider two Lagrangians $L \in C^\infty(E)$, $L' \in C^\infty(E')$ and $\Phi\colon E \to E'$ a morphism of Lie algebroids such that $L' \circ \Phi = L$.

Then, the action functionals $S$ on $\mathcal{P}(J, E)$ and $S'$ on $\mathcal{P}(J, E')$ are related by $\hat{\Phi}$, that is

$$S' \circ \hat{\Phi} = S.$$

**Theorem 3 (Reconstruction)** *Let $\Phi\colon E \to E'$ be a morphism of Lie algebroids. Consider a Lagrangian $L$ on $E$ and a Lagrangian $L'$ on $E'$ such that $L = L' \circ \Phi$. If $a$ is an $E$-path and $a' = \Phi \circ a$ is a solution of Lagrange's equations for $L'$ then $a$ itself is a solution of Lagrange's equations for $L$.*

**Theorem 4 (Reduction)** *Let $\Phi\colon E \to E'$ be a fiberwise surjective morphism of Lie algebroids. Consider a Lagrangian $L$ on $E$ and a Lagrangian $L'$ on $E'$ such that $L = L' \circ \Phi$. If $a$ is a solution of Lagrange's equations for $L$ then $a' = \Phi \circ a$ is a solution of Lagrange's equations for $L'$.*

**Theorem 5 (Reduction by stages)** *Let* $\Phi_1 \colon E \to E'$ *and* $\Phi_2 \colon E' \to E''$ *be fiberwise surjective morphisms of Lie algebroids. Let* $L$, $L'$ *and* $L''$ *be Lagrangian functions on* $E$, $E'$ *and* $E''$, *respectively, such that* $L' \circ \Phi_1 = L$ *and* $L'' \circ \Phi_2 = L'$. *Then the result of reducing first by* $\Phi_1$ *and later by* $\Phi_2$ *coincides with the reduction by* $\Phi = \Phi_2 \circ \Phi_1$.

# Lagrange Multipliers Method

$U$ and $V$ Banach manifolds
$G\colon U \to V$ a submersion
$C = \{\, u \in U \mid G(u) = c \,\}$ submanifold ($c \in V$)
$F\colon U \to \mathbb{R}$ smooth function.

We look for the critical points of $F$ subjected to the constraint $G(u) = c$, that is, the critical points of the restriction of $F$ to the submanifold $C$.

**Theorem 6** *The function $F$ has a critical point at $u_0 \in C$ constrained by $G(u) = c$ if and only if there exists $\lambda \in T_c^*V$ such that $dF(u_0) + \lambda \circ T_cG = 0$.*

# In our problem

$q\colon \mathcal{A}(J,E) \to \boldsymbol{G}$ the quotient map $a \mapsto [a]$
Fix $g \in \boldsymbol{G}$ with source $\boldsymbol{s}(g) = m_0$ and target $\boldsymbol{t}(g) = m_1$. We select an $E$-homotopy class as the set $q^{-1}(g)$.

Difficult to solve. Simplify by fixing an endpoint.

Manifold

$$\mathcal{A}(J,E)_{m_0} = \{\, a \in \mathcal{A}(J,E) \mid \tau(a(t_0)) = m_0 \,\}.$$

Smooth map

$$p\colon \mathcal{A}(J,E)_{m_0} \to \boldsymbol{s}^{-1}(m_1) \qquad p(a) = L_{g^{-1}}(q(a)).$$

Constraint $\quad p(a) = \boldsymbol{\epsilon}(m_1).$

**Proposition 1** *The tangent map to $p$ at $a \in p^{-1}(\boldsymbol{\epsilon}(m_1))$ is*

$$T_a p(\Xi_a(\sigma)) = \sigma(t_1)$$

*for every $\sigma \in \mathrm{Sec}_\gamma(E)$ such that $\sigma(t_0) = 0$.*
*It follows that $p$ is a surjective submersion.*

**Theorem 7** *Let $S_{m_0}$ be the restriction of the action functional $S$ to the submanifold $\mathcal{A}(J, E)_{m_0}$. An admissible curve $a \in \mathcal{A}(J, E)_{m_0}$ is a solution of Lagrange equations if and only if there exists $\lambda \in E^*_{\tau(a(t_1))}$ such that $dS_{m_0}(a) + \lambda \circ T_a p = 0$.*

The multiplier $\lambda$ is related with the momenta defined by the Lagrangian:
the equation $\langle dS(a), v\rangle + \langle \lambda, T_a p(v)\rangle$ for $v = \Xi_a(\sigma)$, reads

$$\int_{t_0}^{t_1} \langle \delta L(\dot{a}(t)), \sigma(t)\rangle = \langle \theta_L(a(t_1)) - \lambda, \sigma(t_1)\rangle.$$

For a solution $a$ of Lagrange equations, $\delta L(\dot{a}(t)) = 0$, and since $\sigma(t_1)$ is arbitrary, we have that

$$\lambda = \theta_L(a(t_1)).$$

# Conclusions

- The Euler-Lagrange equations for a Lagrangian system on a Lie algebroid are the equations for the critical points of the action functional defined on the space of $E$-paths on the Lie algebroid.
- Variations are curves in a manifold (infinite dimensional) and the action is stationary.
- It is not a variational principle of nonholonomic type, it is neither a vakonomic principle.

- Reduction by a symmetry group does not destroy the variational character of the problem.
- One can also obtain such equations by using Lagrange multiplier method (at least in the integrable case), but not in the heuristic classical form.
- The variational character of the equations of motion has nothing to do with the integrability of the Lie algebroid by a Lie groupoid.