
Control by Interconnection of Physical Systems

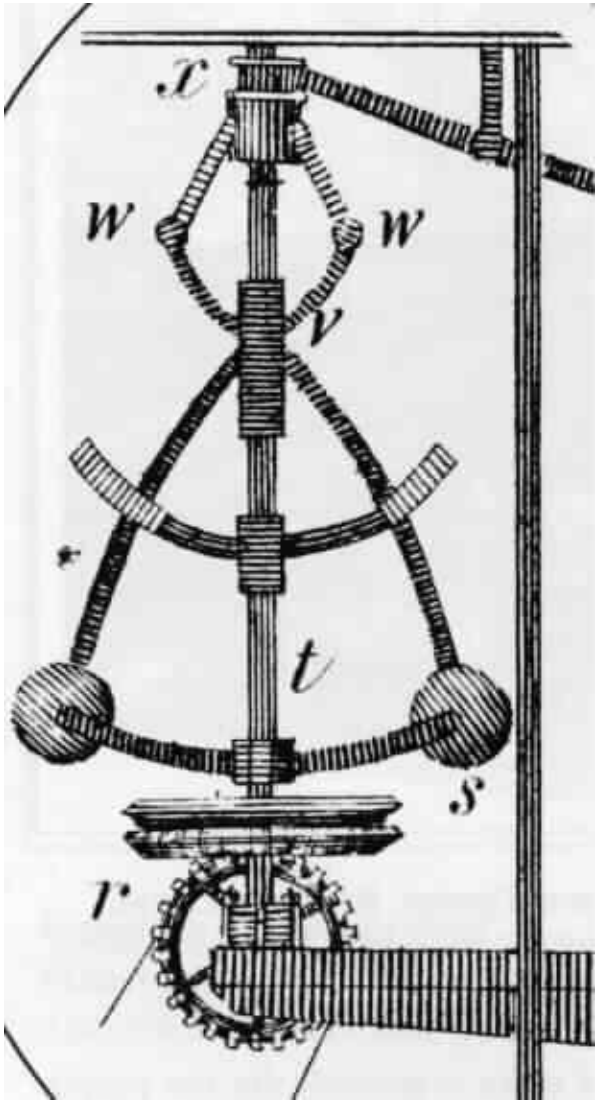
Romeo Ortega

Laboratoire des Signaux et Systèmes, Supélec

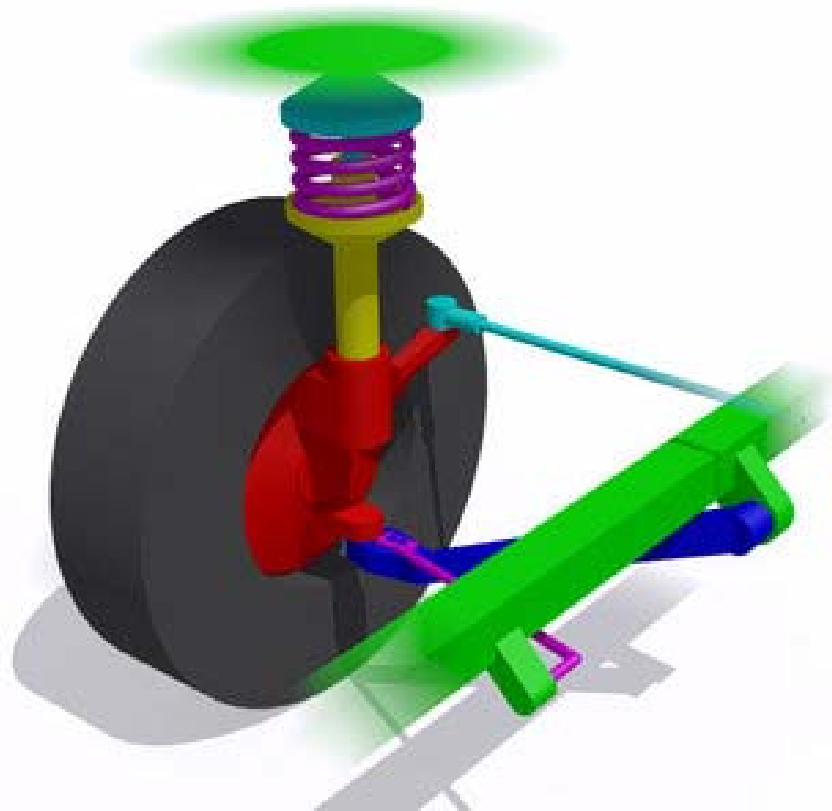
Gif-sur-Yvette, France

Main Message: Provide a new paradigm, alternative to signal-processing, for control of physical systems.

Controllers by Interconnection are as Old as Control Itself



They're Pervasive and Efficient

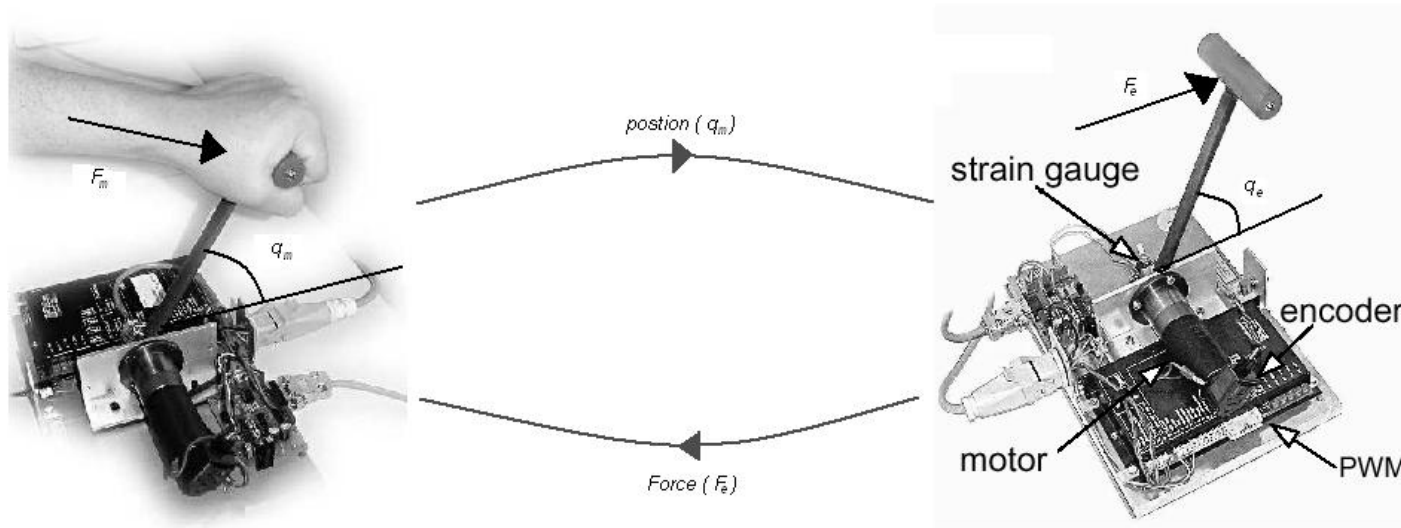


Even in your Privy

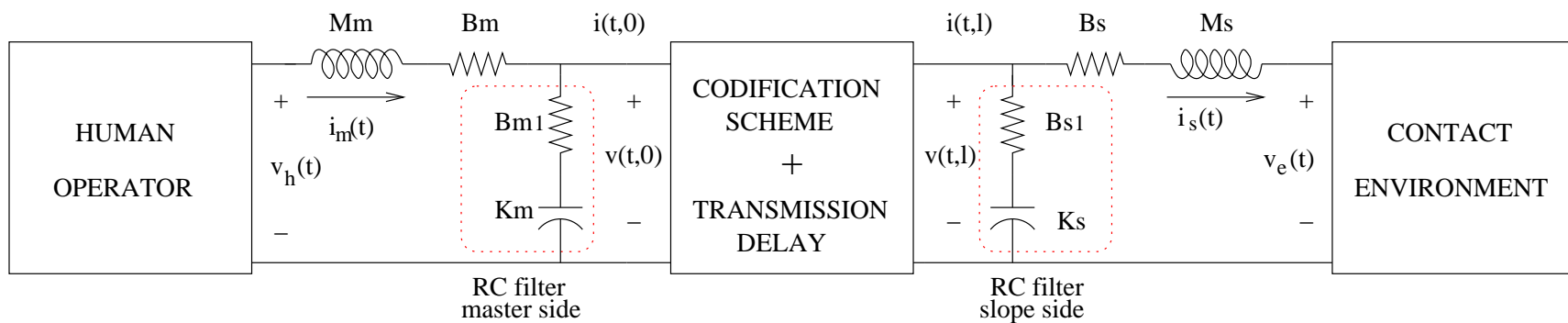


Natural Way to Represent Interactions with Environment

Two mechanical systems, a human-controlled **master** and a **teleoperated slave**

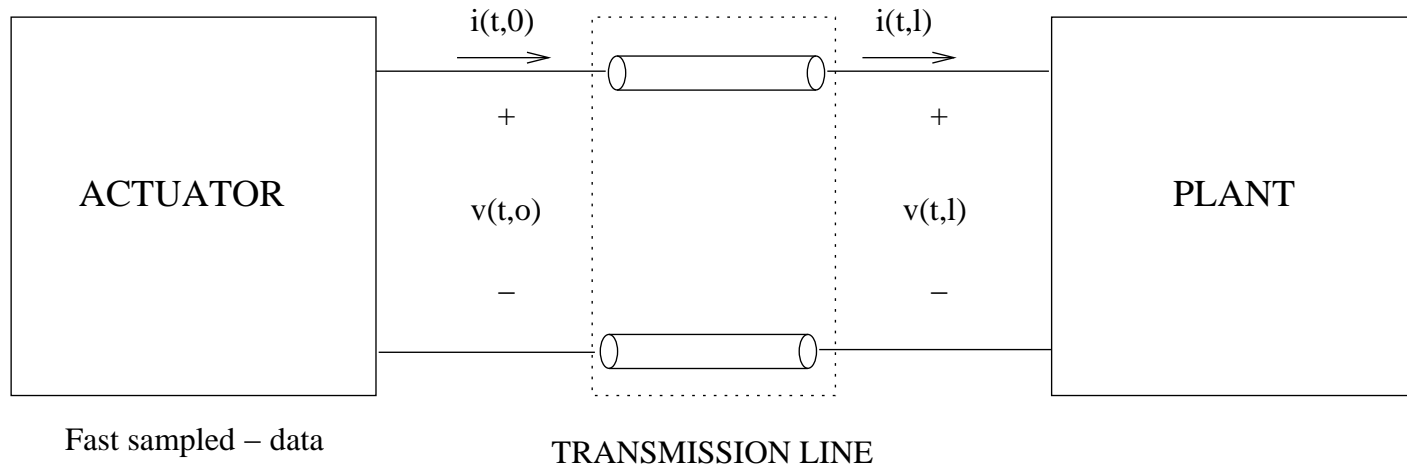


(Anderson/ Spong, '89): Transforming delays into transmission lines

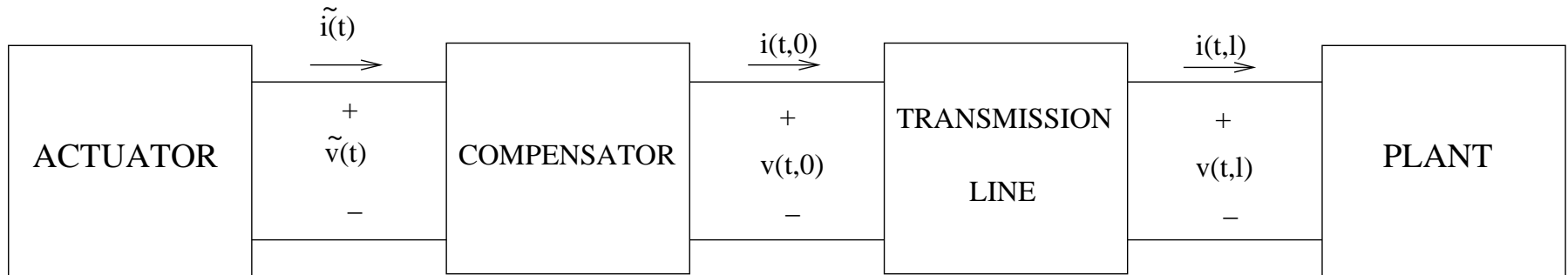


Sometimes the Only Solution

Overvoltage Problem The presence of **long cables** between a **fast-sampling** actuator and plant induces oscillations: The cables behave like a **transmission line**.



(Ortega/Spong, US Patent 07): vice versa



♡ See also Wednesday's talk!

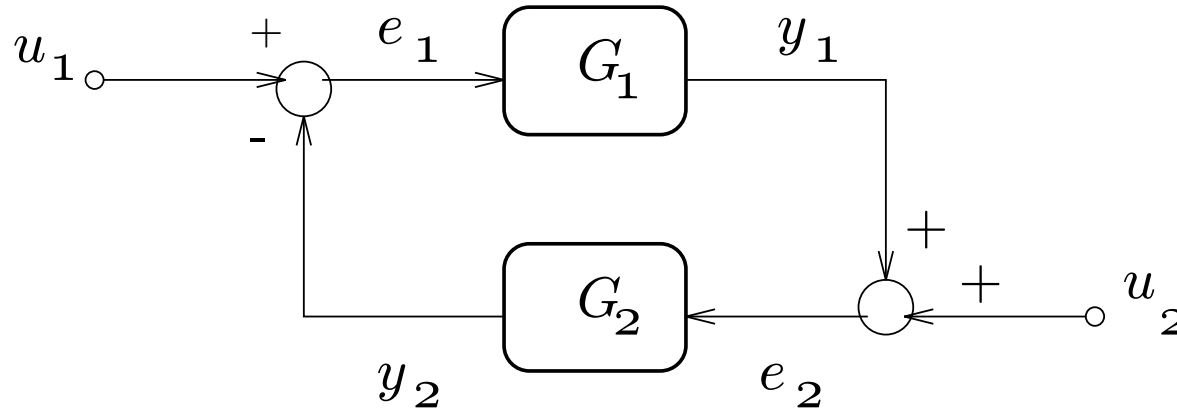
Works Even if We Don't Know Why!

(Spong'06): Synchronization by interconnection, an open problem since the 17th century.



Main Message: Paradigm Shift for Controller Design

Classical formulation: Signal-processing viewpoint



- System model and controller are signal processors: $G_1 : e_1 \rightarrow y_1$, $G_2 : e_2 \rightarrow y_2$.
- Control specifications in terms of signals: tracking, disturbance attenuation, etc.
- Uncertainty represented via the “ $\Sigma - \Delta$ paradigm”:
 - discriminated via **filtering**,
 - very successful for linear time-invariant (LTI) systems
- Control computed from solution of Riccati eqs (\mathcal{H}_∞ , \mathcal{H}_2 -designs).
- “Impossible” in nonlinear case:
 - nonlinear systems “mix” the frequencies,
 - far from obvious computations (NL filtering, Hamilton-Jacobi-Bellman PDE).

Passivity–Based Control: An Energy–Processing Viewpoint

- View plant as **energy–transformation**, as opposed to signal–transformation, multiport device
- Consider systems that satisfy (generalized) energy–conservation:
$$\text{Stored energy} = \text{Supplied energy} + \text{Dissipation}$$
- Control objective in PBC: preserve the energy–conservation property but with **desired** energy and dissipation functions

$$\text{Desired stored energy} = \text{New supplied energy} + \text{Desired dissipation}$$

In other words

$$\text{PBC} = \text{Energy Shaping} + \text{Damping Assignment}$$

- Two possible formulations:
 - State feedback (also called Standard PBC)
 - Control by Interconnection (Cbl)
- Objectives of these lectures:
 - Provide a unified framework
 - Explore the relations between the two formulations

Analytical vs Computational Approaches to Control

- Modern analytical (model-based) control theory is **not** providing solutions to practical control problems with “**strong nonlinearities**” (phenomena that cannot be captured with linear models)
- Existing **analytical** designs rely on **high gain**,
 - E.g. backstepping, sliding modes, Lyapunov domination
 - Intrinsically conservative
 - Amplifies noise
 - Energy consumption...
- Trend in **applications** (prevailing?): (black-box, data-based) **computational** “solutions”
 - Expand NL on a basis + some kind of NL inversion
 - Neural networks, fuzzy controllers, genetic algorithms, etc
 - They might work but we will not understand **why/when?**
 - How to select the fuzzyfication–de-fuzzyfication rules?
 - How many neuron layers? Training?

“New” Computational Trend

- Approach
 - Approximating NL by a (large number of) “linear terms” (piece-wise, LPV,...)
 - Postulating an optimization problem (usually leading to LMI’s)
 - Feasibility checked with particular **numerical** cases
- Questions
 - Why a performance criterion (with constraints) captures the control objective?
 - Wasn’t the **fragility** of optimal control the starting point for robust control?
 - Weighting coefficient selection Gordian knot? Is the addition of receding horizons useful?
 - Computational complexity? (Possible for slow systems with “monotonic” behaviors, e.g. process control)
- Analytical vs computational approaches to control
 - Is the objective of control theory to **generate code** that "solves the problem"?
 - What do we learn about the system by doing this?
 - “Control theory = Number crunching” is a reductionist view
 - **NL analysis** leads to an understanding of the systems behavior
 - Control action best understood adopting a systems **interconnection viewpoint**

Passivity–Based Control Programme

- Consider models that capture main **physical** ingredients:
 - energy, dissipation and interconnection
 - Port–Hamiltonian (PH) systems
- Attain classical control objectives (stability, performance) as **by–products** of:
 - energy–shaping,
 - interconnection modification and
 - damping assignment.

Applications of PBC

- Mass–balance systems, electrical motors, magnetic levitation systems, power systems, power converters, underwater vehicles, surface vessels, (air)spacecrafts, walking robots, bilateral teleoperation, underactuated mechanical systems....

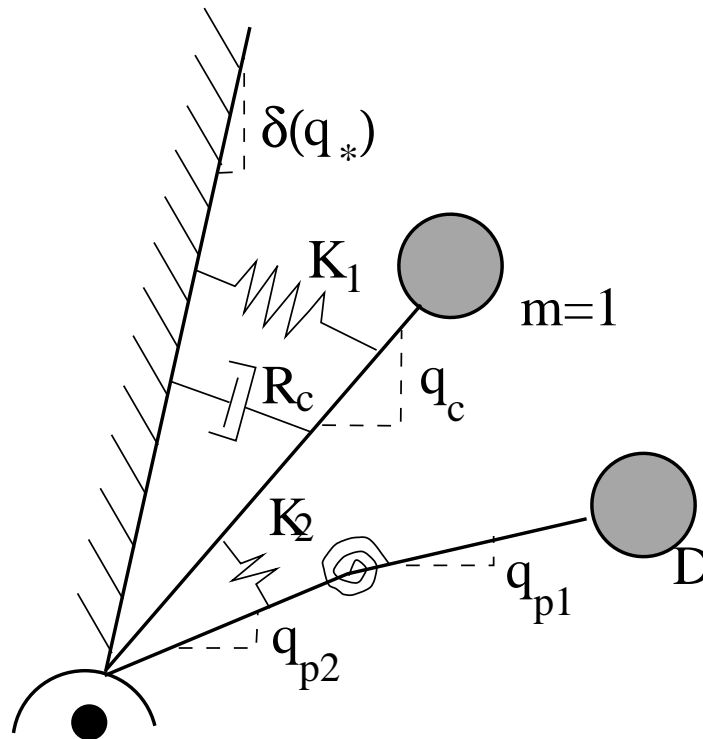
Advantages of PBC

Advantages of energy–shaping (over nonlinearity cancellation and **high gain**)^a

- Handle on **performance**, not just stability
- Respect, and effectively exploit, the structure of the system to
 - incorporate **physical** knowledge,
 - provide physical interpretations to the control action
- Energy serves as a **lingua franca** to communicate with practitioners
- There's an elegant **geometrical** characterization of
 - power–conserving interconnections (via Dirac structures) and
 - passifiable NL systems (in terms of stable invertibility and relative degree)

^aEuphemistically called “nonlinearity domination”.

Example: Control by Interconnection of a Flexible Pendulum



● Plant energy: $H(q_p, p_p) = \frac{1}{2} p_p^\top D^{-1}(q_p) p_p + V(q_p)$

● Controller energy:

$$H_c(q_c, p_c, q_{p2}) = \frac{1}{2} |p_c|^2 + \frac{1}{2} (q_c - q_{p2})^\top K_2 (q_c - q_{p2}) + \frac{1}{2} (q_c - \delta)^\top K_1 (q_c - \delta)$$

● Controller Rayleigh dissipation function: $\frac{1}{2} \dot{q}_c^\top R_c \dot{q}_c$

Energy Shaping Applications

Transient Performance of the Ball and Beam

- Steep energy function \Rightarrow quick response but bad transient \heartsuit
- "Wider" energy function \Rightarrow slower response without overshoot transient \heartsuit
- No regulation of transient excursions \Rightarrow ball gets off the bar \heartsuit
- Limiting level sets \Rightarrow ball remains in the bar \heartsuit

Vertical Take-off and Landing Aircraft

- Providing tuning knobs to decide a maneuver \heartsuit

Smooth Swing-up of Underactuated Mechanical Systems

- Acrobot ♡
- Furuta's Pendulum ♡

Orbital Stabilization of Passive Walking Robot

- Problem cannot be posed in terms of tracking, a natural approach is to **regulate the kinetic-potential energy exchange** ♡
- 'Achieved assigning an energy function like a "Mexican sombrero" ♡

Basic References

● (Material of the course)

- R. Ortega, A. van der Schaft, F. Castaños and A. Astolfi: Control By State–Modulated Interconnection Of Port–Hamiltonian Systems, *NOLCOS 07*. (To appear *IEEE Trans. Automat. Contr.*).
- R. Ortega, A. van der Schaft, I. Mareels and B. Maschke: Putting Energy Back In Control, *IEEE Control Syst. Magazine*, Vol. 21, No. 2, April 2001, pp. 18–33.
- R. Ortega and E. Garcia-Canseco: Interconnection and Damping Assignment Passivity–Based Control: A Survey, *European J of Control*, Vol. 10, pp. 432–450, 2004.
- R. Ortega, M. Spong, F. Gomez and G. Blankenstein: Stabilization Of Underactuated Mechanical Systems Via Interconnection and Damping Assignment, *IEEE Trans. Automat. Contr.*, Vol. AC–47, No. 8, 2002, pp. 1218–1233.

● (General I/O theory)

- A. J. van der Schaft: *L_2 –Gain and Passivity Techniques in Nonlinear Control*, Springer–Verlag, Berlin, 1999.

● (PBC of Euler–Lagrange systems)

- R. Ortega, A. Loria, P. J. Nicklasson and H. Sira–Ramirez: *Passivity–based control of Euler–Lagrange systems*, Springer-Verlag, Berlin, Sept. 1998.

Layout

1. Cyclo-passivity and formulation of PBC stabilization problem
2. Basic control by interconnection (Cbl) of Port-Hamiltonian systems
 - Energy-Casimir method
 - Dissipation obstacle
3. Extensions of Cbl method
 - Generating new cyclo-passivity properties
 - Overcoming the dissipation obstacle
 - Control by state-modulated interconnection
4. Standard (State-feedback) PBC
 - Energy balancing control (EBC)
 - Interconnection and damping assignment (IDA)
 - Power shaping
5. Comparison of the two methods
 - Domain of applicability
 - Standard PBC as a projection of Cbl
6. Conclusions and outlook

List of Acronyms

BIDA	Basic IDA
CBI	Control by interconnection
CBI_{PS}	CBI with power shaping output
CBI_{PS}^{SM}	CBI with power shaping output and state modulated interconnection
EBC	Energy–balancing control
IDA	Interconnection and damping assignment
NL	Nonlinear
PBC	Passivity–based control
PDE	Partial differential equation
PS	Power shaping
y_{PS}	Power shaping output

1. Key Property: Cyclo–Passivity

Definition We say that the m –port system with state $x \in \mathbb{R}^n$, and power port variables $u, y \in \mathbb{R}^m$

$$\Sigma : \begin{cases} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{cases}$$

is **cyclo–passive** if there exists storage (energy) function $H : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\underbrace{H[x(t)] - H[x(0)]}_{\text{stored energy}} \leq \underbrace{\int_0^t u^\top(s)h(x(s))ds}_{\text{supplied energy}}$$

If $H(x) \geq 0$ then the system is **passive** with port variables (u, y) and storage function $H(x)$.

Remark For passive systems we have

$$-\int_0^t u^\top(s)y(s)ds \leq H[x(0)] < \infty \quad \Rightarrow$$

amount of energy that can be extracted from a passive system is **bounded**.

Stabilization via Energy Shaping and Damping Injection

With $u(t) \equiv 0$, we have

$$H[x(t)] \leq H[x(0)] \quad \Rightarrow$$

- Trajectories tend to converge towards points of **minimum energy**
- If the minima are **strict** $H(x)$ qualifies as a Lyapunov function for them
- To operate the system around some desired equilibrium point, say x_* , PBC shapes the energy to assign a strict minimum at this point.
- Furthermore, if we terminate the port with

$$u = -K_{di}y, \quad K_{di} = K_{di}^\top > 0$$

we get

$$\dot{H} \leq -y^\top K_{di}y \leq 0.$$

Hence, $x(t) \rightarrow 0$ if $h(x)$ is **detectable** (for the closed-loop system). That is, if

$$h(x(t)) \equiv 0 \Rightarrow x(t) \rightarrow 0.$$

Port–Hamiltonian (PH) Systems

- PH model of a physical system

$$\Sigma_{(u,y)} : \begin{cases} \dot{x} &= [\mathcal{J}(x) - \mathcal{R}(x)]\nabla H + g(x)u \\ y &= g^\top(x)\nabla H \end{cases}$$

- $u^\top y$ has units of **power** (voltage–current, speed–force, angle–torque, etc.)
 - $\mathcal{J} = -\mathcal{J}^\top$ is the interconnection matrix, specifies the internal power–conserving structure (oscillation between potential and kinetic energies, Kirchhoff’s laws, transformers, etc.)
 - $\mathcal{R} = \mathcal{R}^\top \geq 0$ damping matrix (friction, resistors, etc.)
 - g is input matrix.
- PH systems are cyclo–passive

$$\dot{H} = -\nabla H^\top \mathcal{R} \nabla H + u^\top y.$$

- **Invariance of PH structure** Power preserving interconnection of PH systems is PH.
- Nice geometric structure formalized with notion of **Dirac** structures.
- Most nonlinear cyclo–passive systems can be written as PH systems. Actually, in (network) modeling is the other way around!

Examples: RLC Circuits

- For any (possibly nonlinear) LC circuit we have

$$\dot{x} = \begin{bmatrix} 0 & \Gamma \\ -\Gamma^\top & 0 \end{bmatrix} \nabla H + gu, \quad y = g^\top \nabla H$$

where $x = \text{col}(q_C, \phi_L)$, $H = H_E(q_C) + H_M(\phi_L)$ – electric plus magnetic energies, Γ comes from Kirchhoff's laws and u are (external) voltage and current sources.

- Example: LTI Series RLC circuit

- Total energy,

$$H(x) = \frac{1}{2C} x_1^2 + \frac{1}{2L} x_2^2$$

- Co-energy variables $\nabla H = \text{col}(v_C, i_L)$,

- PH model, u voltage source

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -R \end{bmatrix}}_{\mathcal{J}-\mathcal{R}} \underbrace{\begin{bmatrix} \frac{x_1}{C} \\ \frac{x_2}{L} \end{bmatrix}}_{\nabla H} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_g u, \quad y = \nabla_{x_2} H = \frac{x_2}{L} = i_L$$

Mechanical Systems

● State $x = \text{col}(q, p)$, $p \triangleq M(q)\dot{q}$ momenta.

● Total energy:

$$H(q, p) = \frac{1}{2}p^\top M^{-1}(q)p + U(q)$$

● Assuming linear friction,

$$F = R\dot{q}, \quad R = R^\top \geq 0$$

● PH model, u forces/torques

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & I \\ -I & -R \end{bmatrix} \nabla H + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u \\ y &= \nabla_p H = M^{-1}p \quad (= \dot{q}) \end{aligned}$$

G input matrix (actuated coordinates).

Electromechanical Systems

● Assuming linear magnetics, i.e., $\phi = L(\theta)i \in \mathbb{R}^n$, $L(\theta) = L^\top(\theta) \geq 0$, one mechanical d.o.f., $\theta \in \mathbb{R}$, voltages $u \in \mathbb{R}^m$.

● State $x = \text{col}(\phi, \theta, p)$, $p = m\dot{\theta}$.

● Total energy:

$$H(x) = \frac{1}{2}\phi^\top L^{-1}(\theta)\phi + \frac{1}{2m}p^2 + U(\theta)$$

● Co-energy variables $\nabla H = \text{col}(i, -\tau, \dot{\theta})$, where τ force (torque) of electrical origin.

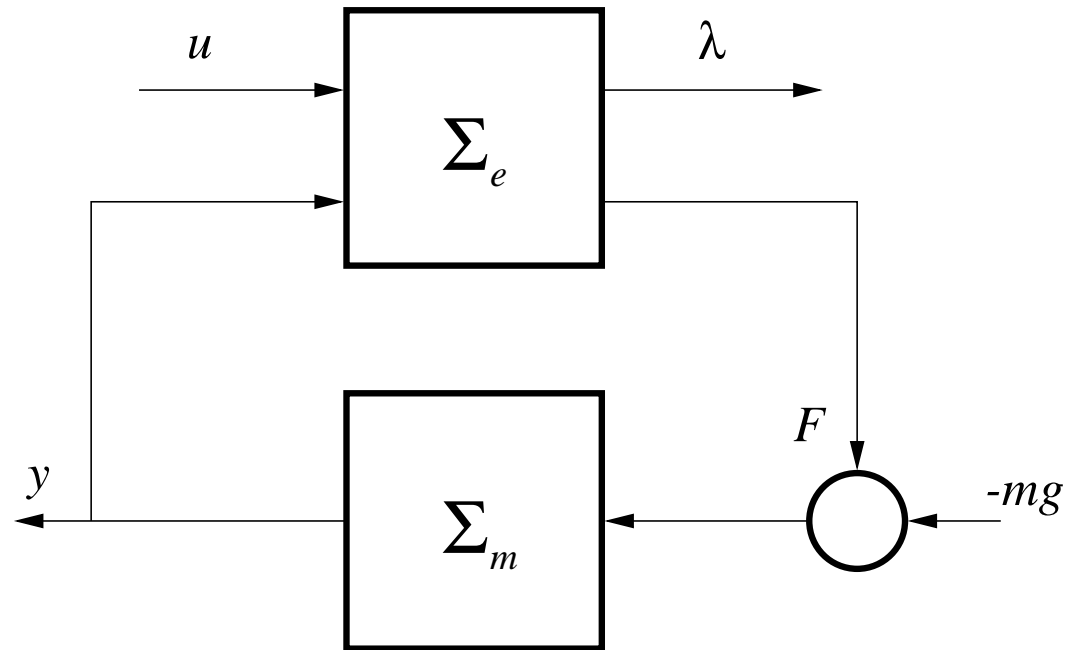
● PH model

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -R & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \nabla H + \begin{bmatrix} M & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ -\tau_L \end{bmatrix} \\ y &= \text{col}(Mi, \omega), \end{aligned}$$

$\tau_L \in \mathbb{R}$ load torque, $M \in \mathbb{R}^{n \times m}$ defines actuated coordinates.

Passive Sub-Systems Feedback Decomposition

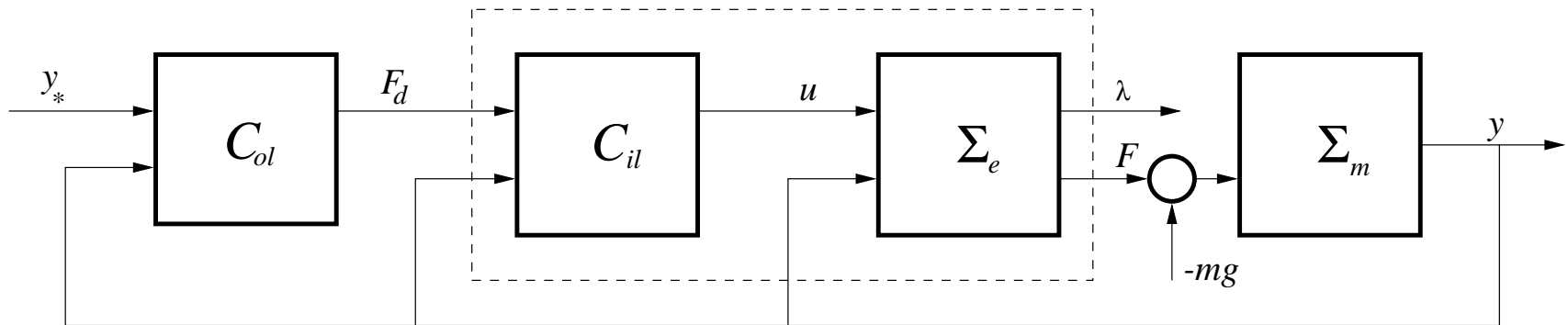
- The maps $\Sigma_e : (u, \dot{\theta}) \rightarrow (i, F)$ and $\Sigma_m : (F - mg) \rightarrow \dot{\theta}$ are **passive** (with storage function $H_e \geq 0$, $\frac{m}{2} \dot{\theta}^2 \geq 0$, resp.)



- Designing a PBC that “views” Σ_m as a passive disturbance, suggests a nested-loop control configuration

Classical Nested-Loop Controller

- Often employed in applications, where the inner-loop is designed “neglecting” the mechanical part. This is rationalized via time-scale separation arguments.



- Passivity provides a rigorous formalization of this approach, without this assumption, see (Ortega et al's Book, '98).

Power Converters

- More general class of PH models:

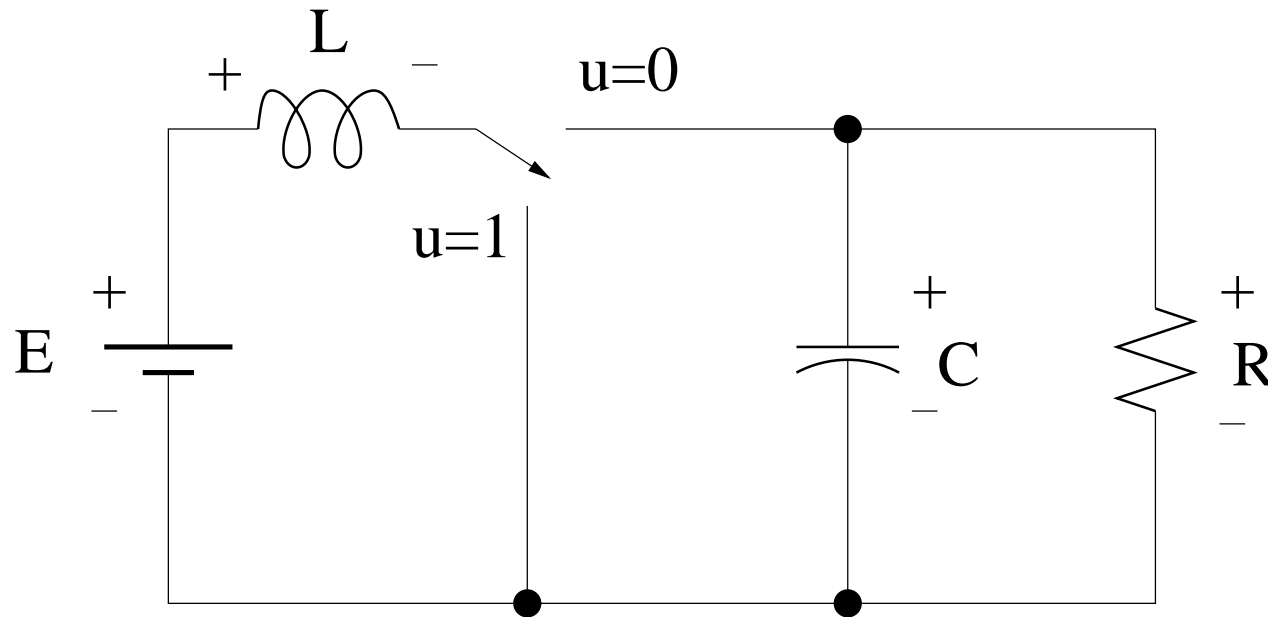
$$\dot{x} = [\mathcal{J}(x, u) - \mathcal{R}(x)]\nabla H + g(x, u)$$

- The control u modifies the interconnection and input matrices
- Assuming: fast switching, u is the duty cycle.
- State $x = \text{col}(\phi_L, q_C)$
- For linear L_i, C_i the total energy is

$$H(x) = \frac{1}{2}x_1^\top L^{-1}x_1 + \frac{1}{2}x_2^\top C^{-1}x_2,$$

where $L = \text{diag}\{L_i\}, C = \text{diag}\{C_i\}$.

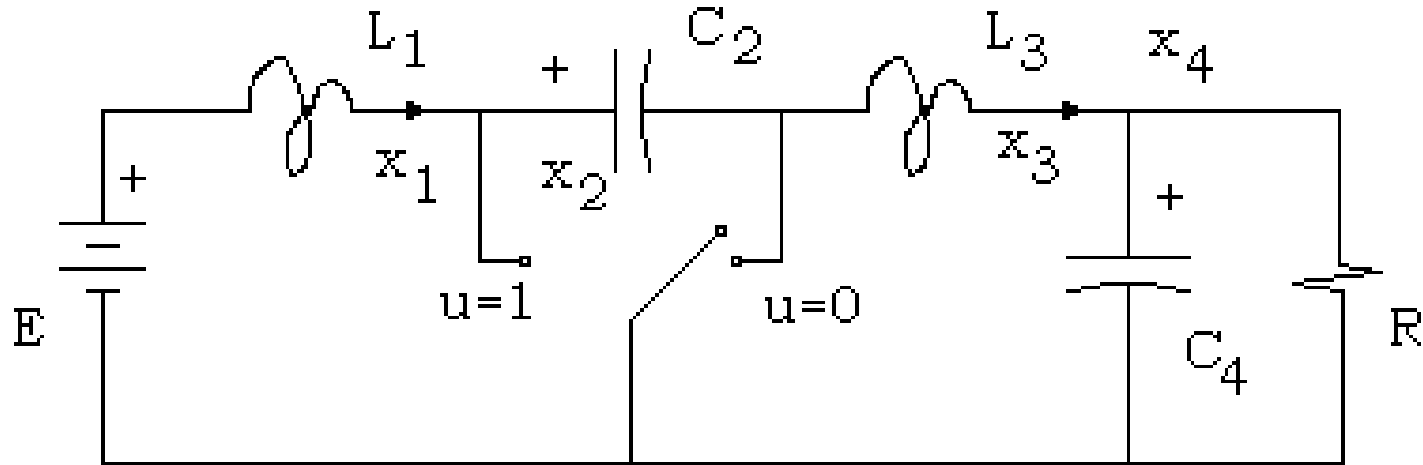
Boost Converter



PH Model: $x = \text{col}(\phi_L, q_C)$

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & -u \\ u & \frac{-1}{R} \end{bmatrix}}_{\mathcal{J}(u) - \mathcal{R}} \nabla H + \begin{bmatrix} E \\ 0 \end{bmatrix}$$

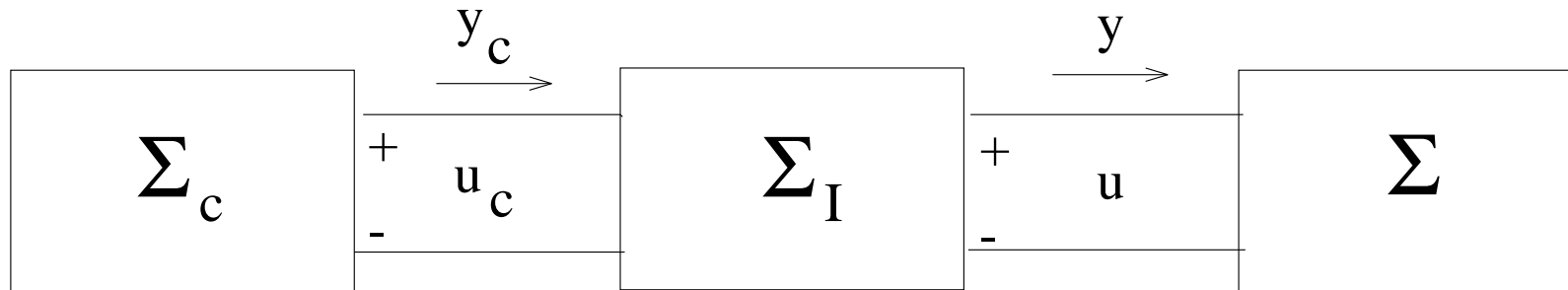
Cuk Converter



PH Model

$$\dot{x} = \begin{bmatrix} 0 & -(1-u) & 0 & 0 \\ 1-u & 0 & u & 0 \\ 0 & -u & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{R} \end{bmatrix} \nabla H + \begin{bmatrix} E \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

2. Control by Interconnection (Cbl)



- Plant (Σ) and controller (Σ_c), with states $x \in \mathbb{R}^n, \zeta \in \mathbb{R}^m$, are **cyclo-dissipative**, that is, $\exists H : \mathbb{R}^n \rightarrow \mathbb{R}, H_c : \mathbb{R}^m \rightarrow \mathbb{R}$, such that

$$\dot{H} \leq u^\top y, \quad \dot{H}_c \leq u_c^\top y_c.$$

- Interconnection subsystem (Σ_I) is power-preserving (lossless):

$$y^\top u + y_c^\top u_c = y^\top v \quad (\Leftarrow u = -y_c + v, u_c = y).$$

- Interconnected system satisfies $\dot{H} + \dot{H}_c \leq v^\top y \Rightarrow H + H_c$ is the **new energy**.

Problem

Although $H_c(\zeta)$ is free, not clear how to “shape” x ?

Preliminaries: Invariant Sets (Manifolds)

Definition Consider the dynamical systems $\dot{x} = f(x)$, $x \in \mathbb{R}^n$. The set $\mathcal{M} \subset \mathbb{R}^n$ is invariant (w.r.t. the flow of the system) iff

$$x(0) \in \mathcal{M} \Rightarrow x(t) \in \mathcal{M}, \quad \forall t \geq 0.$$

Fact Given $D : \mathbb{R}^n \rightarrow \mathbb{R}^{n-m}$, $D \in \mathcal{C}^1$. The set $\mathcal{M} \triangleq \{x \in \mathbb{R}^n \mid D(x) = 0\}$ is invariant iff

$$\dot{D}|_{x \in \mathcal{M}} = 0.$$

Furthermore, the level sets (foliation) $\mathcal{M}_\kappa \triangleq \{x \in \mathbb{R}^n \mid D(x) = \kappa\}$, $\kappa \in \mathbb{R}$ are invariant iff

$$\dot{D} = 0 \quad (\heartsuit).$$

Remarks

- For LTI systems, $\dot{x} = Ax$, the set spanned by the (real) eigenvectors of A is invariant.
- (\heartsuit) is a PDE: $\nabla D f = 0$.
- The functions $D(x)$ are called conserved quantities (dynamical invariants).

Energy–Casimir Method

Proposition Assume, $\exists \mathcal{C} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that the level sets $\Omega_\kappa \triangleq \{(x, \zeta) | \zeta = \mathcal{C}(x) + \kappa\}$ are invariant, for all $\kappa \in \mathbb{R}$. Then, for all $\Phi : \mathbb{R}^m \rightarrow \mathbb{R}$, the function

$$W(x, \zeta) \triangleq H(x) + H_c(\zeta) + \Phi(\mathcal{C}(x) - \zeta).$$

satisfies

$$\dot{W} \leq v^\top y$$

That is, the system is cyclo–passive w.r.t. $W(x, \zeta)$, which (given \mathcal{C}) can be **shaped** selecting H_c and Φ .

Proof Invariance of Ω_κ is equivalent to

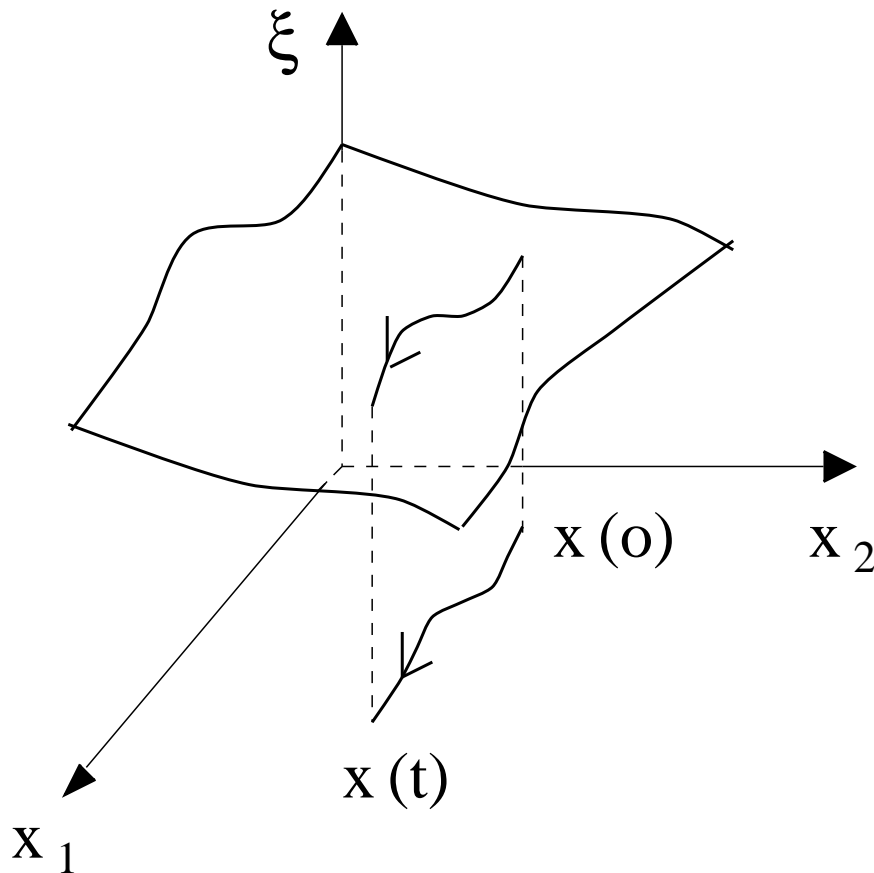
$$\dot{\zeta} - \frac{d}{dt}\mathcal{C}(x) = 0.$$

Hence,

$$\dot{\Phi} = \nabla\Phi(\dot{\mathcal{C}} - \dot{\zeta}) = 0 \Rightarrow \dot{W} = \dot{H} + \dot{H}_c.$$

◇ ◇ ◇

Caveat: Achieving Asymptotic Stability



All level sets

$$\Omega_{\kappa} = \{(x, \zeta) | \zeta = \mathcal{C}(x) + \kappa\},$$

$\kappa \in \mathbb{R}$ are invariant. In principle, we must set

$$\zeta(0) = \zeta_{\star} + \mathcal{C}(x(0)) - \mathcal{C}(x_{\star})$$

to ensure that the trajectory starts (and remains) in $\Omega_{\kappa_{\star}}$, with

$$\kappa_{\star} \triangleq \zeta_{\star} - \mathcal{C}(x_{\star}),$$

that contains the desired equilibrium. A more practical solution is to **estimate** κ , adding an integrator.

Basic Cbl for PH Systems

- Given a PH system,

$$\Sigma_{(u,y)} \begin{cases} \dot{x} &= F(x)\nabla H(x) + g(x)u \\ y &= g^\top(x)\nabla H(x), \end{cases} \Rightarrow \dot{H} \leq u^\top y$$

where we defined $F(x) := \mathcal{J}(x) - \mathcal{R}(x)$, $\mathcal{J} = -\mathcal{J}^\top$, $\mathcal{R} = \mathcal{R}^\top \geq 0$.

- PH controller (nonlinear integrators), $\zeta \in \mathbb{R}^m$

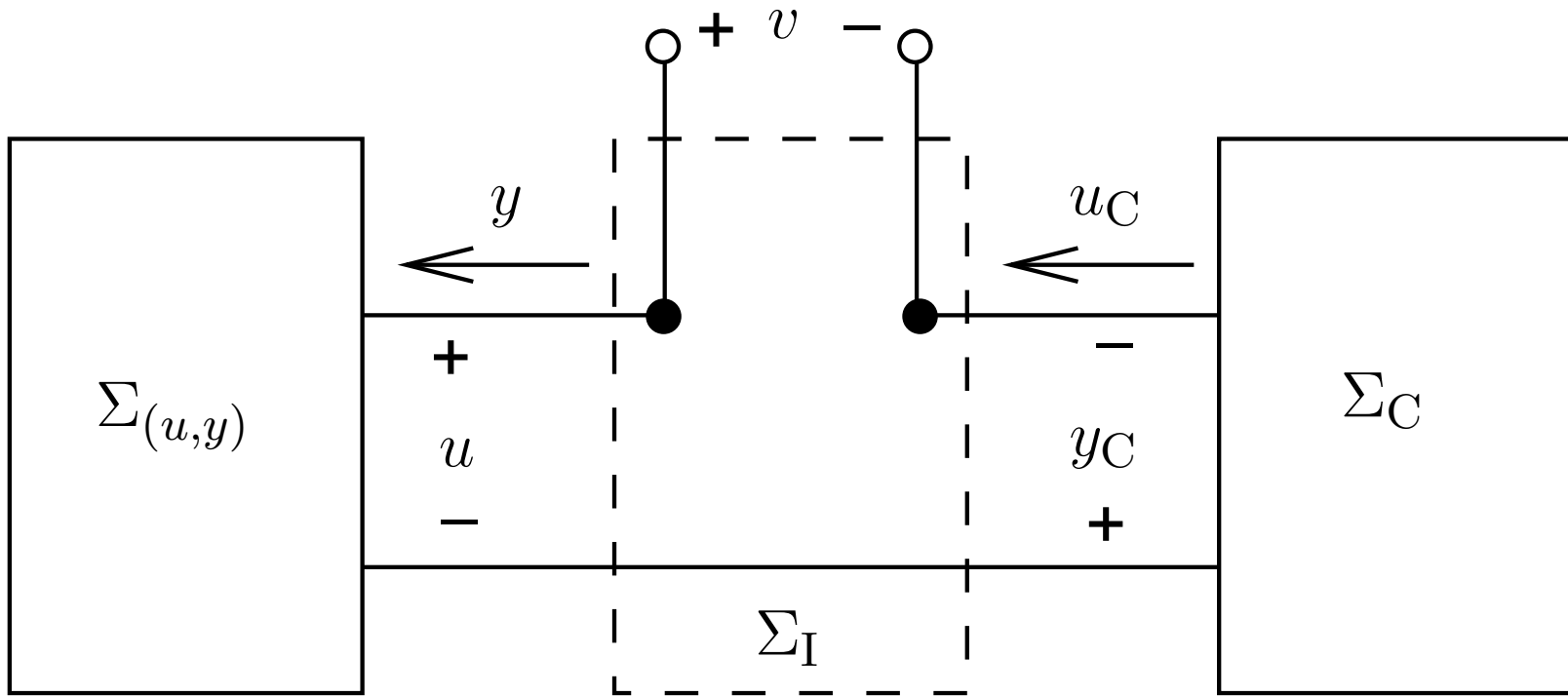
$$\Sigma_c : \begin{cases} \dot{\zeta} &= u_c \\ y_c &= \nabla_\zeta H_c(\zeta), \end{cases} \Rightarrow \dot{H}_c = u_c^\top y_c$$

- Standard negative feedback interconnection

$$\Sigma_I : \begin{cases} \begin{bmatrix} u \\ u_c \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ y_c \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix} \end{cases} \Rightarrow \dot{H} + \dot{H}_c \leq v^\top y$$

- For ease of presentation, and **with** loss of generality, we have taken $\zeta \in \mathbb{R}^m$.

Negative Feedback Interconnection Subsystem Σ_I



Casimir Functions for PH Systems

- Recalling the total energy function

$$W(x, \zeta) = H(x) + H_c(\zeta) + \Phi(\mathcal{C}(x) - \zeta),$$

where H is given, \mathcal{C} will be computed and H_c, Φ selected to shape W .

- We want \mathcal{C} to be **independent** of H and H_c – these are called **Casimir functions**.
- The dynamics of the interconnected system is given by

$$\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} F & -g \\ g^\top & 0 \end{bmatrix} \begin{bmatrix} \nabla H \\ \nabla H_c \end{bmatrix} + \begin{bmatrix} g \\ 0 \end{bmatrix} v.$$

We are looking for \mathcal{C} such that $\dot{\mathcal{C}} - \dot{\zeta} = 0$ **for all** H and H_c . Thus, we get the PDEs

$$\begin{bmatrix} (\nabla \mathcal{C})^\top & -I_m \end{bmatrix} \begin{bmatrix} F & -g \\ g^\top & 0 \end{bmatrix} = 0.$$

- Note that $(\nabla \mathcal{C})^\top g = 0$ ensures $\dot{\mathcal{C}} - \dot{\zeta} = 0$ even with $v \neq 0$.

Conditions for CbI

Proposition Assume there exists a vector function $\mathcal{C} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$$\begin{bmatrix} F^\top \\ g^\top \end{bmatrix} \nabla \mathcal{C} = \begin{bmatrix} g \\ 0 \end{bmatrix} \quad (\text{CbI} - \text{PDE})$$

Then, **for all** functions $\Phi : \mathbb{R}^m \rightarrow \mathbb{R}$, the following cyclo-passivity inequality is satisfied

$$\dot{W} \leq v^\top y,$$

where the shaped storage function $W : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is defined as

$$W(x, \zeta) \triangleq H(x) + H_c(\zeta) + \Phi(\mathcal{C}(x) - \zeta).$$

Proof

$$\dot{\Phi} = \nabla \Phi(\dot{\mathcal{C}} - \dot{\zeta}) = 0.$$

◇ ◇ ◇

The Dissipation Obstacle

Proposition If (Cbl-PDE) admits a solution then

$$\mathcal{R}\nabla_x \Phi(\mathcal{C}(x) - \zeta) = 0,$$

for all $\Phi : \mathbb{R}^m \rightarrow \mathbb{R}$. Consequently, energy cannot be shaped for coordinates that are affected by physical damping.

Proof (Cbl-PDE) \Leftrightarrow

$$\begin{aligned} F^\top \nabla \mathcal{C} = g, \quad g^\top \nabla \mathcal{C} = 0 &\Rightarrow (\nabla \mathcal{C})^\top F^\top \nabla \mathcal{C} = 0 \\ &\Rightarrow \mathcal{R}\nabla \mathcal{C} = 0 \end{aligned}$$

Proof completed with $\nabla_x \Phi = \nabla \mathcal{C} \nabla \Phi$. ◇◇◇

Remarks

- OK for mechanical systems where dissipation enters in the momenta equations—that need not be shaped.
- Note that $\mathcal{J}\nabla \mathcal{C} = -g$. Hence, (Cbl-PDE) is equivalent to

$$F\nabla \mathcal{C} = -g, \quad g^\top \nabla \mathcal{C} = 0.$$

The Dissipation Obstacle and Power Balance

- We proved that (Cbl-PDE) $\Rightarrow F\nabla\mathcal{C} = -g, \mathcal{R}\nabla\mathcal{C} = 0$. Hence,

$$\mathcal{R}(F^\top F)^{-1}F^\top g = 0,$$

which is a **necessary** condition for the existence of Casimirs.

- Denote $(\cdot)(x_*) := (\cdot)_*$, where x_* is an equilibrium to be stabilized and assume (CBI-PDE) holds. From $\dot{x} = F\nabla H + gu$ we have that

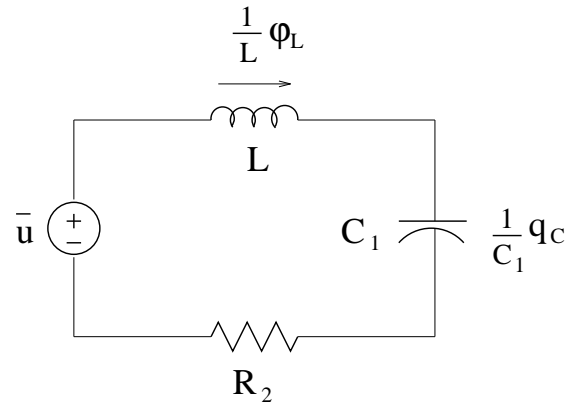
$$\begin{aligned} 0 = F_*\nabla H_* + g_*u_* &\Rightarrow \nabla H_* = -(F_*^\top F_*)^{-1}F_*^\top g_*u_* \\ \Rightarrow \mathcal{R}\nabla H_* = -\mathcal{R}(F_*^\top F_*)^{-1}F_*^\top g_*u_* &\Rightarrow \mathcal{R}\nabla H_* = 0 \Rightarrow \nabla^\top H_*\mathcal{R}_*\nabla H_* = 0. \end{aligned}$$

- From the power balance equation

$$\dot{H} = -\nabla H^\top \mathcal{R}\nabla H + u^\top y$$

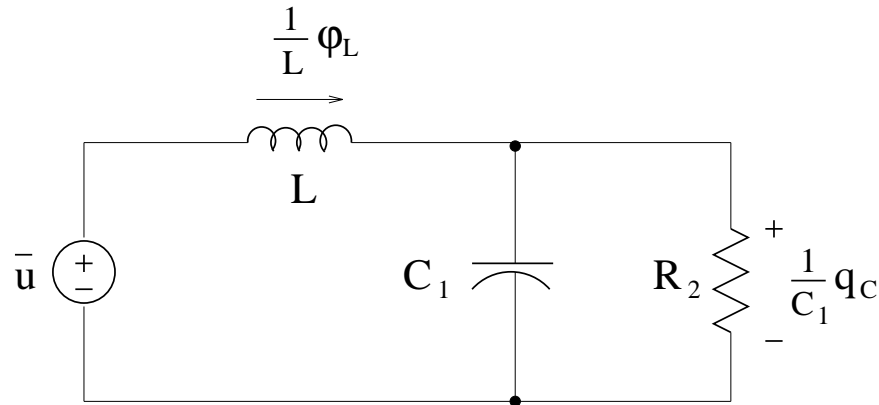
we see that, if the system admits a Casimir then the dissipated power at the equilibrium should be zero, i.e., $u_*^\top y_* = 0$. That is, we should be able to stabilize the system extracting a **finite amount of energy from the source**.

Example without Pervasive Dissipation



- Energy function $H(q_C, \phi_L) = H_E(q_C) + H_M(\phi_L)$. Assume $0 = \arg \min H_M$.
- Co-energy variables $\nabla H = \text{col}(v_C, i_L)$
- PH model
$$\begin{bmatrix} \dot{q}_C \\ \dot{\phi}_L \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -R_2 \end{bmatrix} \nabla H + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
- Equilibria: $\nabla_{\phi_L} H(\bar{\phi}_L) = 0 \Rightarrow \bar{\phi}_L = 0 \Rightarrow$ no need to shape H_M
 - **No** dissipation in the coordinate to be shaped (q_C)
 - (CBI-PDE): $F \nabla \mathcal{C} = -g, g^\top \nabla \mathcal{C} = 0$, a Casimir is $\mathcal{C}(q_C, \phi_L) = q_C$, thus we can add to $H + H_c$ an arbitrary function $\Phi(q_C - \zeta) \Rightarrow$ stabilizable via Cbl
- Power balance: $\dot{H} = -R_2 (\nabla_{\phi_L} H)^2 + u \nabla_{\phi_L} H \Rightarrow$ **dissipation zero at equilibria.**

Example with Pervasive Dissipation



- Same energy function, but the dissipation has changed: $\mathcal{J} - \mathcal{R} = \begin{bmatrix} \frac{-1}{R_2} & 1 \\ -1 & 0 \end{bmatrix}$
- Equilibria: $(\nabla_{q_C} H, \nabla_{\phi_L} H) = (u_*, R_2 u_*) \neq (0, 0) \Rightarrow$ Dissipation in a coordinate to be shaped (q_C)
- Not stabilizable via Cbl!
- We have $\mathcal{R}F^{-1}g = \text{col}(\frac{-1}{R_2}, 0)$. Therefore, the necessary condition for the existence of Casimirs, $\mathcal{R}F^{-1}g = 0$, is not satisfied
- Power balance: $\dot{H} = -\frac{1}{R_2} (\nabla_{q_C} H)^2 + u \nabla_{\phi_L} H \Rightarrow$ dissipation **not zero** at equilibria.

3. Generating New Cyclo–Passivity Properties

Idea To find new cyclo–passive outputs look for full rank matrices $F_d : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$, with

$$F_d + F_d^\top \leq 0 \quad (SYM)$$

and storage functions $H_{PS} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$F \nabla H = F_d \nabla H_{PS}.$$

Proposition For all solutions F_d of the PDE

$$\nabla \left(F_d^{-1} F \nabla H \right) = \left[\nabla \left(F_d^{-1} F \nabla H \right) \right]^\top \quad (PO - PDE)$$

verifying (SYM) there exists a storage function H_{PS} such that

$$\Sigma_{(u, y_{PS})} \begin{cases} \dot{x} & = F \nabla H + gu \\ y_{PS} & = -g^\top F_d^{-\top} (F \nabla H + gu) \end{cases} \Rightarrow \dot{H}_{PS} \leq u^\top y_{PS}.$$

◇ ◇ ◇

Proof

Poincare's Lemma Given $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $f \in \mathcal{C}^1$. There exists $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\nabla\psi = f$ if and only if

$$\nabla f = (\nabla f)^\top.$$

◇ ◇ ◇

Consequently,

$$(PO - PDE) \quad \Leftrightarrow \quad \nabla H_{\text{PS}} = F_d^{-1} F \nabla H,$$

We then have the following chain of implications

$$\begin{aligned} F_d \nabla H_{\text{PS}} = F \nabla H &\Rightarrow \dot{x} = F_d \nabla H_{\text{PS}} + gu \\ &\Leftrightarrow F_d^{-1} \dot{x} = \nabla H_{\text{PS}} + F_d^{-1} gu \\ &\Rightarrow \dot{x}^\top F_d^{-1} \dot{x} = \dot{H}_{\text{PS}} + \dot{x}^\top F_d^{-1} gu \\ &\Rightarrow 0 \geq \dot{H}_{\text{PS}} + \dot{x}^\top F_d^{-1} gu \\ &\Leftrightarrow y_{\text{PS}}^\top u \geq \dot{H}_{\text{PS}}, \end{aligned}$$

where we used $\dot{x}^\top F_d^{-1} \dot{x} \leq 0$.

◇ ◇ ◇

Remarks

- If F is full rank $F_d = F$ solves (PO–PDE). In this case, $H_d = H$ and we obtain the new power–balance equation

$$\dot{H} = \dot{x}^\top F^{-1} \dot{x} + u^\top y_{\text{PS}}.$$

Comparing with $\dot{H} = u^\top y + \nabla^\top H F \nabla H$, we see that the new passive output is obtained **swapping the damping**.

- The new cyclo–passive output y_{PS} is equal to $g^\top \nabla H_{\text{PS}}$ if and only if the dissipation obstacle for the PH system with port variables $(u, g^\top \nabla H_{\text{PS}})$ is **absent**, that is

$$\mathcal{R}_d F_d^{-1} g = 0 \quad \Leftrightarrow \quad y_{\text{PS}} = -g^\top F_d^{-\top} (F_d \nabla H_{\text{PS}} + g u) = g^\top \nabla H_{\text{PS}}.$$

- Setting $F_d = F$ in (PO–PDE) we obtain

$$\mathcal{R} F^{-1} g = 0 \Leftrightarrow y = y_{\text{PS}}.$$

- For (underactuated) mechanical systems $\mathcal{R} F^{-1} g = 0$.

Solving (PO–PDE)

Proposition (Ortega et al. '04) For all matrices $M : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$, with $M(x) = M^\top(x)$ and all $\lambda \in \mathbb{R}$, such that

$$\tilde{M} \triangleq \frac{1}{2} [(\nabla^2 H)M + \nabla(M\nabla H) + 2\lambda I_n]$$

is full rank,

$$F_d^{-1} = \tilde{M}F^{-1}$$

solves (PO–PDE). The resulting storage function being

$$H_{PS} = \lambda H + (\nabla H)^\top M \nabla H.$$

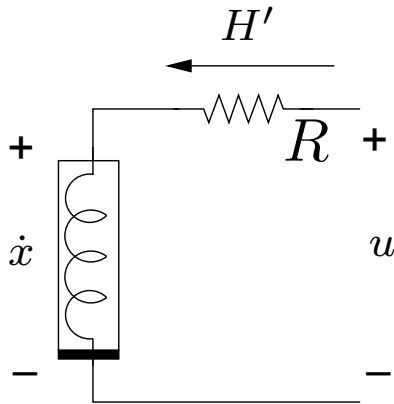
◇ ◇ ◇

Remarks

- Generates a family of solutions of (PO–PDE) parameterized in terms of (M, λ) .
- Case of constant M reported in (Brayton/Moser '64)

An Example of Generation of y_{PS}

Consider the nonlinear RL circuit with x the inductor flux



energy function the magnetic energy stored in the inductor, i.e.,

$$H(x) = \int_0^x i(s) ds ,$$

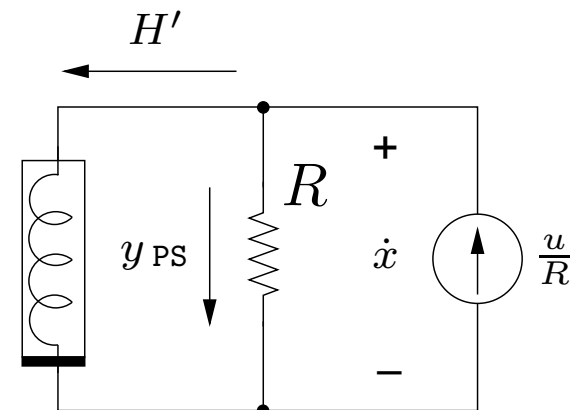
and **passive output** the electrical current

$$y = H'(x) = i.$$

The new system is

$$\Sigma(u, y_{PS}) : \begin{cases} \dot{x} &= -RH' + u \\ y_{PS} &= -H' + \frac{1}{R}u \end{cases}$$

which has the physical realization



that is obtained applying the classical **Thevenin–Norton** transformation to the original circuit.

Cbl with $\Sigma_{(u, y_{PS})}$ Overcomes the Dissipation Obstacle

Proposition Assume (PO–PDE) admits a solution F_d verifying (SYM) and such that

$$F_d \nabla \mathcal{C} = -g \quad (CbI_{PS} - PDE)$$

for some vector function $\mathcal{C} : \mathbb{R}^n \rightarrow \mathbb{R}^m$. Consider the PH system $\Sigma_{(u, y_{PS})}$ coupled with the PH controller Σ_c through the power–preserving interconnection subsystem

$$\Sigma_I^{PS} : \begin{cases} \begin{bmatrix} u \\ u_c \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{PS} \\ y_c \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix}. \end{cases}$$

Then, for all functions $\Phi : \mathbb{R}^m \rightarrow \mathbb{R}$, the following cyclo–passivity inequality is satisfied

$$\dot{W}_{PS} \leq v^\top y_{PS},$$

where

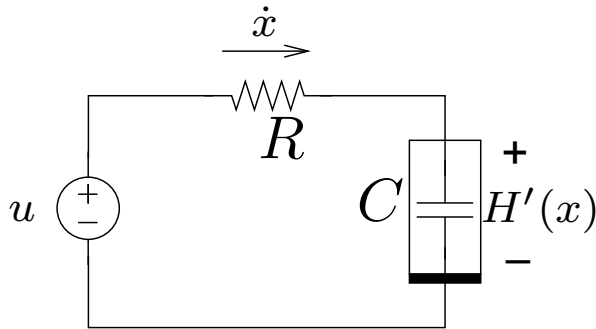
$$W_{PS}(x, \zeta) \triangleq H_{PS}(x) + H_c(\zeta) + \Phi(\mathcal{C}(x) - \zeta),$$

with $H_{PS} = \int (F_d^{-1} F \nabla H) dx$.

Remark The condition $g^\top \nabla \mathcal{C} = 0$ is **absent**.

An Example of Stabilization via CbI_{PS}

Consider the nonlinear RC circuit with x the capacitor charge, $\dot{x} = i$ and $H' = v$



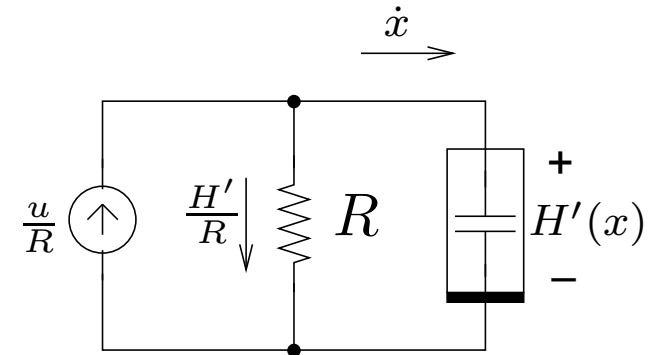
One PH model

$$\begin{aligned}\dot{x} &= -\frac{1}{R}H' + \frac{1}{R}u \\ y &= \frac{1}{R}H' .\end{aligned}$$

Power balance equation

$$\dot{H} = -\frac{1}{R}(H')^2 + H' \frac{1}{R}u .$$

Best understood from the equivalent circuit:



obtained applying the **Thevenin–Norton** transformation A more physically sensible way of viewing the system is $u \rightarrow \dot{x}$, that is,

$$\Sigma_{(u, y_{PS})} : \begin{cases} \dot{x} &= -\frac{1}{R}H' + \frac{1}{R}u \\ y_{PS} &= -\frac{1}{R}H' + \frac{1}{R}u = \dot{x}, \end{cases}$$

cont'd

- There is no \mathcal{C} that satisfies the Casimir conditions: $-\frac{1}{R}\mathcal{C}' = \frac{1}{R}$, $\frac{1}{R}\mathcal{C}' = 0$, so this problem can not be solved using CbI .
- For any x_* , the corresponding $u_* = (H')_*$ and $y_* = \frac{1}{R}(H')_*$ yield the power

$$u_* y_* = \frac{1}{R} (H')_*^2.$$

Hence, $u_* y_* = 0$, necessary for EB-PBC, can only be satisfied if the capacitor voltage at the equilibrium is zero.

- $\mathcal{R}F^{-1}g = -\frac{1}{R} \neq 0$ implies that $y \neq y_{PS}$, so we will try the CbI_{PS} technique.
- The Casimirs for CbI_{PS} are solutions of $FC' = -g$, which becomes $\frac{1}{R}\mathcal{C}' = \frac{1}{R}$, so $\mathcal{C}(x) = x$ will be a Casimir.
- Look for functions H_c, Φ such that the function

$$W_{PS}(x, \zeta) = H(x) + H_c(\zeta) + \Phi(x - \zeta)$$

has an isolated minimum at a given equilibrium point $(x_*, 0)$. For simplicity, we took $\zeta_* = 0$.

cont'd

- Compute

$$(\nabla W_{\text{PS}})_{\star} = 0 \quad \Leftrightarrow \quad \begin{cases} \Phi'_{\star} & = & -u_{\star} \\ (H'_c)_{\star} & = & -u_{\star} \end{cases} .$$

The Hessian $\nabla^2 W_{\text{PS}} > 0$ if and only if

$$\begin{aligned} \Phi'' &> -H'' \\ H_c'' &> \frac{-H'' \Phi''}{H'' + \Phi''} . \end{aligned}$$

- Fix a quadratic

$$H_c = \frac{1}{2C_c} (\zeta - C_c u_{\star})^2,$$

with $C_c \in \mathbb{R}_+$.

- For Φ we propose $\Phi(z) = \frac{\beta}{2} z^2 + \gamma z,$

$$\beta x_{\star} + \gamma = -u_{\star}.$$

- Evaluating minima conditions) at the equilibrium turns into

$$\begin{aligned}\beta &> -H''_{\star} \\ \frac{1}{C_c} &> -\frac{H''_{\star} \beta}{H''_{\star} + \beta},\end{aligned}$$

from where it is easy to see that if $H''_{\star} > 0$, we can take $\beta = 0$ and the equilibrium $(x_{\star}, 0)$ will be stable, for all $C_c > 0$, with Lyapunov function $W_{\text{PS}}(x, \zeta) - (W_{\text{PS}})_{\star}$, where

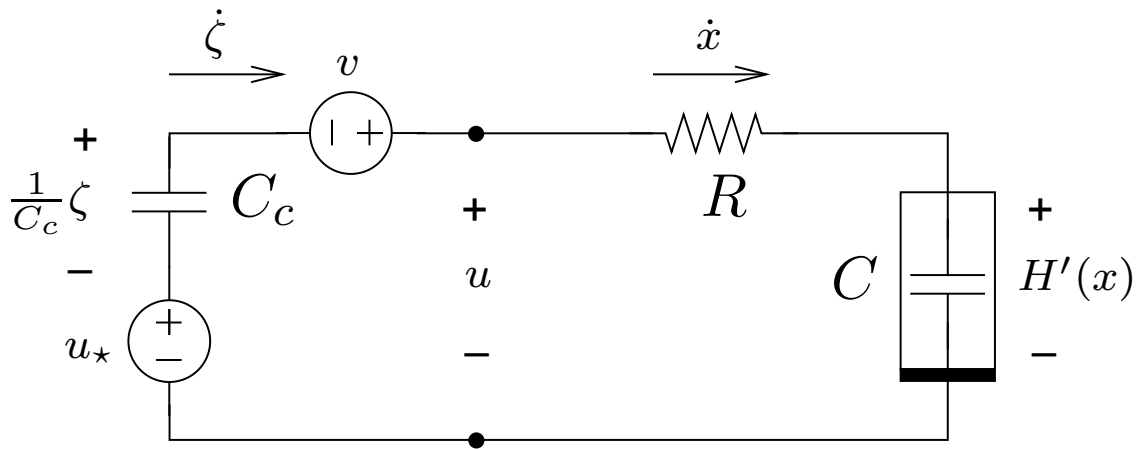
$$W_{\text{PS}}(x, \zeta) = H(x) + \frac{1}{2C_c} (\zeta - C_c u_{\star})^2 - u_{\star} (x - \zeta).$$

- The controller is given by

$$\Sigma_c + \Sigma_I^{\text{PS}} : \begin{cases} \dot{\zeta} &= \frac{1}{R} (-H' + u_{\star} - \frac{1}{C_c} \zeta + v) \\ u &= u_{\star} - \frac{1}{C_c} \zeta + v. \end{cases}$$

cont'd

Physical realization



Remark It can be implemented without distinction of "inputs" and "outputs".

Control by State–Modulated Interconnection with $\Sigma_{(u,y)}$

Proposition Assume the PDE

$$\begin{bmatrix} g^\perp F^\top \\ g^\top \end{bmatrix} \nabla \mathcal{C} = 0, \quad (CbI^{SM} - PDE)$$

admits a solution for some vector function $\mathcal{C} : \mathbb{R}^n \rightarrow \mathbb{R}^m$. The PH system $\Sigma_{(u,y)}$ with the PH controller Σ_c and the state–modulated power–preserving interconnection subsystem

$$\Sigma_I^{SM} : \begin{cases} \begin{bmatrix} u \\ u_c \end{bmatrix} = \begin{bmatrix} 0 & -\alpha(x) \\ \alpha^\top(x) & 0 \end{bmatrix} \begin{bmatrix} y \\ y_c \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix}, \end{cases}$$

where $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ is defined as

$$\alpha = -(g^\top g)^{-1} g^\top F \nabla \mathcal{C}.$$

Then, for all functions $\Phi : \mathbb{R}^m \rightarrow \mathbb{R}$, $\dot{W} \leq v^\top y$, where

$$W(x, \zeta) \triangleq H(x) + H_c(\zeta) + \Phi(\mathcal{C}(x) - \zeta).$$

Control by State–Modulated Interconnection with $\Sigma_{(u, y_{PS})}$

Proposition Assume (PO–PDE) admits a solution F_d verifying (SYM) and such that

$$g^\perp F_d \nabla C = 0, \quad (CbI_{PS}^{SM} - PDE)$$

for some vector function $C : \mathbb{R}^n \rightarrow \mathbb{R}^m$, where $g^\perp \in \mathbb{R}^{(n-m) \times n}$ is a full rank left annihilator of g , that is, $g^\perp g = 0$ and $\text{rank } g^\perp = n - m$. The PH system $\Sigma_{(u, y_{PS})}$ with the PH controller Σ_c and the state–modulated power–preserving interconnection subsystem

$$\Sigma_I^{SM} : \begin{cases} \begin{bmatrix} u \\ u_c \end{bmatrix} = \begin{bmatrix} 0 & -\alpha(x) \\ \alpha^\top(x) & 0 \end{bmatrix} \begin{bmatrix} y_{PS} \\ y_c \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix}, \end{cases}$$

where $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ is defined as

$$\alpha = -(g^\top g)^{-1} g^\top F_d \nabla C.$$

Then, for all functions $\Phi : \mathbb{R}^m \rightarrow \mathbb{R}$, $\dot{W}_{PS} \leq v^\top y_{PS}$, where

$$W_{PS}(x, \zeta) \triangleq H_{PS}(x) + H_c(\zeta) + \Phi(C(x) - \zeta).$$

Summary of Cbl

● (CbI)

$$\begin{bmatrix} F \\ g^\top \end{bmatrix} \nabla \mathcal{C} = \begin{bmatrix} -g \\ 0 \end{bmatrix}.$$

● (CbISM)

$$\begin{bmatrix} g^\perp F \\ g^\top \end{bmatrix} \nabla \mathcal{C} = 0.$$

● (Basic CbI_{PS})

$$F \nabla \mathcal{C} = -g.$$

● (Basic CbI_{PS}SM)

$$g^\perp F \nabla \mathcal{C} = 0.$$

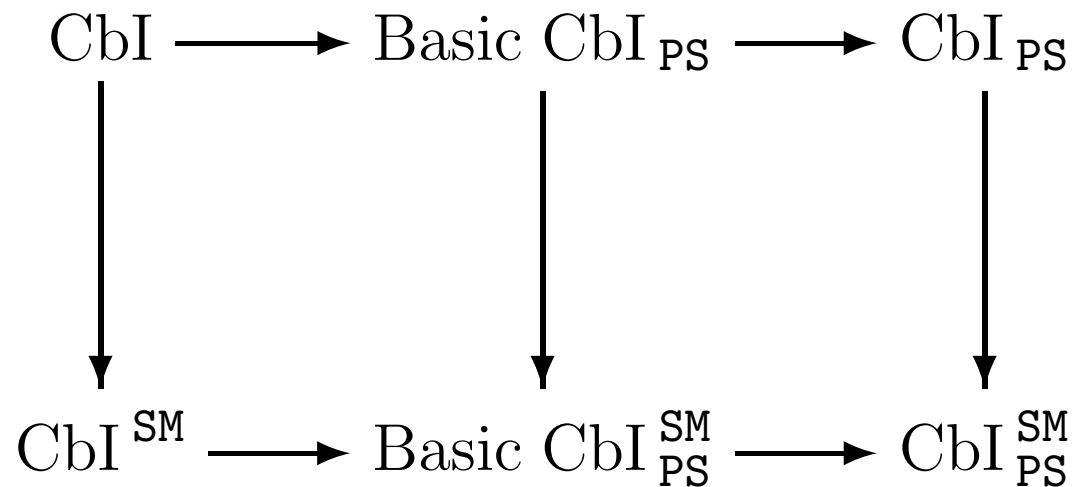
● (CbI_{PS})

$$F_d \nabla \mathcal{C} = -g,$$

● (CbI_{PS}SM)

$$g^\perp F_d \nabla \mathcal{C} = 0,$$

Implication diagram (from the point of view of solvability of the PDEs)



Notation: $A \rightarrow B$ means that the set of solutions of the PDEs of B is strictly larger than the one of A, consequently the set of plants to which B is applicable is also strictly larger.

Also, $A \leftrightarrow B$ if the PDEs are the same.

4. Stabilization of PH Systems via Standard PBC

Passivation Objective Consider the PH system

$$\Sigma_{(u,y)} \begin{cases} \dot{x} &= F\nabla H + gu \\ y &= g^\top \nabla H, \end{cases} \Rightarrow \dot{H} = u^\top y + \underbrace{\nabla H^\top F \nabla H}_{-d \leq 0}$$

with x_* an equilibrium to be stabilized. Select a control action

$$u = \hat{u}(x) + v,$$

so that the closed-loop system satisfies the desired dissipation equality (DDE)

$$\dot{H}_d = v^\top z - d_d \quad (DDE)$$

- $H_d(x)$ has a strict **minimum** at x_* , ($\Rightarrow v \rightarrow z$ is passive)
- $d_d(t) \geq 0$ desired damping, ($\int d_d$ is the dissipated energy), and
- z is the new passive output (to be defined.)

Remark State feedback, for ease of presentation. Must derivations applicable for (f, g, h) systems.

Characterization of Cyclo–Passivity

Proposition (Hill/Moylan'76) Let $H : \mathbb{R}^n \rightarrow \mathbb{R}$, $H \in \mathcal{C}^1$. The system

$$\dot{x} = f(x) + g(x)u, \quad y = h(x)$$

is cyclo–passive with storage function H , i.e. $\dot{H} \leq u^\top y$, **iff** there exists a **damping function** $d : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that

$$\nabla H^\top f = -d \quad (HM1)$$

$$g^\top \nabla H = h \quad (HM2)$$

Proof (\Leftarrow)

$$\dot{H} \leq u^\top y$$

equivalent to

$$\underbrace{\nabla H^\top f}_{\leq 0} + \underbrace{(\nabla H^\top g - h) u}_{=0} \leq 0$$

◇ ◇ ◇

Remark Similar result, but more involved (HM2), for relative degree zero systems.

Satisfying the Desired Dissipation Equality

Corollary Consider $\Sigma_{(u,y)}$ with $u = \hat{u}(x) + v$. Then (DDE) holds iff

$$\begin{aligned}\nabla H_d^\top (F\nabla H + g\hat{u}) &= -d_d & (HM1) \\ z &= g^\top \nabla H_d\end{aligned}$$

Approach

Given (F, g, H) . Select **desired damping** $d_d \geq 0$ to be able to characterize a set of **assignable** energy functions and controls, (H_d, \hat{u}) , that solve (HM1).

Remarks

● For LTI systems, $\dot{x} = Ax + Bu$, with

$$\hat{u} = Kx, \quad H_d = \frac{1}{2}x^\top P_d x, \quad d_d = \frac{1}{2}x^\top R_d x$$

(HM1) becomes the Lyapunov equation $P_d(A + BK) + (A + BK)^\top P_d = -R_d$.

● In (HM1) the data are H, F and g and unknowns H_d, d_d and \hat{u} .

● Relative degree zero outputs do not help because (HM1) is the same.

A Basic Lemma

Let $g \in \mathbb{R}^{n \times m}$, $m < n$ with $\text{rank } g = m$. Define $g^\perp \in \mathbb{R}^{(n-m) \times n}$ as a full rank left annihilator of g , that is, $g^\perp g = 0$ and $\text{rank } g^\perp = n - m$. For any $b \in \mathbb{R}^n$, $\hat{u} \in \mathbb{R}^m$

$$b + g\hat{u} = 0 \quad \Leftrightarrow \quad \begin{cases} g^\perp b & = & 0 \\ \hat{u} & = & -(g^\top g)^{-1} g^\top b \end{cases} \quad (\heartsuit)$$

Proof First, the matrix $\begin{bmatrix} g^\perp \\ g^\top \end{bmatrix} \in \mathbb{R}^{n \times n}$ is full rank. Hence $(\heartsuit) \Leftrightarrow$

$$\begin{bmatrix} g^\perp \\ g^\top \end{bmatrix} (b + g\hat{u}) = 0$$

Proof completed noting that $\text{rank } g^\top g = m$.

Energy Balancing Control

Proposition Let

$$d_d = d = -\nabla H^\top F \nabla H.$$

Denote $\hat{u} = \hat{u}_{\text{EB}}$ and define the **added** energy function $H_a \triangleq H_d - H$.

- All solutions of the PDEs $\begin{bmatrix} g^\perp F^\top \\ g^\top \end{bmatrix} \nabla H_a = 0$ (**EB - PDE**) define assignable energy functions with

$$\hat{u}_{\text{EB}} = -(g^\top g)^{-1} g^\top F^\top \nabla H_a.$$

- The added energy equals the energy supplied by the controller, that is,

$$\dot{H}_a = -y^\top \hat{u}_{\text{EB}}.$$

- EBC suffers from the **dissipation obstacle**. More precisely,

$$(\text{EB} - \text{PDE}) \Rightarrow \mathcal{R} \nabla H_a = 0.$$

Proof

$$\begin{aligned}(HM1) \text{ with } d_d = d &\Leftrightarrow \nabla H_d^\top (F \nabla H + g \hat{u}_{EB}) = \nabla H^\top F \nabla H \\ &\Leftrightarrow \nabla H_d^\top g \hat{u}_{EB} = -\nabla H_a^\top F \nabla H \\ &\Rightarrow \nabla H^\top g \hat{u}_{EB} = -\nabla H_a^\top F \nabla H \quad (\Leftarrow g^\top \nabla H_a = 0) \\ &\Leftrightarrow \nabla H^\top (F^\top \nabla H_a + g \hat{u}_{EB}) = 0.\end{aligned}$$

Applying the lemma to the term in parenthesis we get the solution

$$g^\perp F^\top \nabla H_a = 0, \quad \hat{u}_{EB} = -(g^\top g)^{-1} g^\top F^\top \nabla H_a$$

To establish the energy balancing property we have

$$\begin{aligned}g^\top \nabla H_a = 0 &\Rightarrow z = y \quad (\Leftarrow z = g^\top \nabla H_d) \\ &\Rightarrow \dot{H}_d = y^\top v - d \quad (\Leftarrow d_d = d) \\ &\Rightarrow \dot{H}_a = -y^\top \hat{u}_{EB} \quad (\Leftarrow \dot{H} = y^\top (\hat{u}_{EB} + v) - d)\end{aligned}$$

Finally, from $F^\top \nabla H_a = -g \hat{u}_{EB}$, premultiplying by ∇H_a^\top , and using $g^\top \nabla H_a = 0$ yields

$$\nabla H_a^\top F^\top \nabla H_a = 0 \quad \Leftrightarrow \mathcal{R} \nabla H_a = 0.$$

Application to Mechanical Systems

- EB-PBC are widely popular for potential energy shaping of mechanical systems. In this case

$$x = \begin{bmatrix} q \\ p \end{bmatrix}, \quad H(q, p) = \frac{1}{2} p^\top M^{-1}(q) p + V(q), \quad F = \begin{bmatrix} 0 & I \\ -I & -R \end{bmatrix}, \quad g(q) = \begin{bmatrix} 0 \\ G(q) \end{bmatrix},$$

where (q, p) are the generalized coordinates and momenta, $M = M^\top > 0$ is the inertia matrix, $R = R^\top \geq 0$ is the dissipation due to friction, G is the input matrix and V is the open-loop potential energy.

- The added energy is $H_a(q) = V_d(q) - V(q)$, V_d is the desired potential energy.

- (EB-PDE) $\begin{bmatrix} g^\perp F^\top \\ g^\top \end{bmatrix} \nabla H_a = 0$ becomes

$$G^\perp (\nabla V_d - \nabla V) = 0,$$

known as the potential energy matching equation.

EB-PBC: An Alternative Viewpoint

Given

$$\dot{x} = f(x) + g(x)u, \quad y = h(x).$$

If we can find a vector function $\hat{u} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that the PDE

$$\left(\frac{\partial H_a}{\partial x}(x) \right)^\top [f(x) + g(x)\hat{u}(x)] = -h^\top(x)\hat{u}(x),$$

can be solved for $H_a : \mathbb{R}^n \rightarrow \mathbb{R}$, and $H_d(x) \triangleq H(x) + H_a(x)$, is such that $x_\star = \arg \min H_d(x)$, then, the state-feedback $u = \hat{u}(x)$ is an EBC, i.e., x_\star is stable with Lyapunov function

$$H_d(x) = H(x) - \int_0^t \hat{u}^\top(x(s))h(x(s))ds.$$

Remark: A necessary condition for the solvability of the PDE is

$$f(\bar{x}) + g(\bar{x})\hat{u}(\bar{x}) = 0 \Rightarrow \underbrace{h^\top(\bar{x})\hat{u}(\bar{x})}_{\text{power}} = 0,$$

i.e. extracted power at the equilibrium \bar{x} should be zero.

Example: Pendulum

● Energy $H = \frac{1}{2}\dot{q}^2 - mgl \cos(q)$

$$\ddot{q} + mgl \sin(q) = u$$

$$y = \dot{q}$$

● Objective: stabilize an equilibrium $(\bar{q}, \bar{\dot{q}}) = (q_d, 0)$

● To achieve stabilization via Energy–Balancing we need

$$-\int_0^t \hat{u}[q(s)]\dot{q}(s)ds = H_a[q(t)] + \kappa$$

$$\Leftrightarrow \dot{H}_a = -\hat{u}(q)\dot{q}.$$

Since H_a is **only** function of q , we can assign it with

$$\hat{u}(q) = -\frac{\partial H_a}{\partial q}(q)$$

cont'd

- Let $H_a(q) = \frac{k_p}{2} (q - q_d + \frac{mgl}{k_p} \sin(q_d))^2$, with $k_p > 0$, yielding

$$\hat{u}(q) = -k_p (q - q_d + \frac{mgl}{k_p} \sin(q_d))$$

- New total energy for the passive port (v, \dot{q}) is

$$H_d(q, \dot{q}) = \frac{1}{2} \dot{q}^2 + \frac{k_p}{2} (q - q_d + \frac{mgl}{k_p} \sin(q_d))^2 - mgl \cos(q)$$

which has a minimum in $(q_d, 0)$ if $k_p > mgl$.

- Asymptotic stability with

$$v = -k_{di} \dot{q}, \quad k_{di} > 0$$

- (Takegaki–Arimoto, '81) PD controller!

General Mechanical Systems and Physical Interpretation

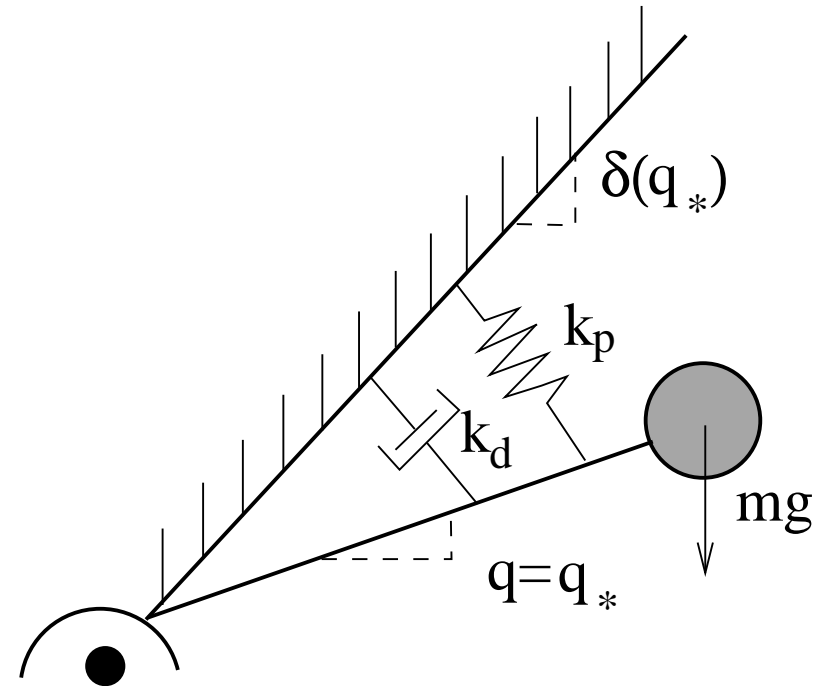
- **Passivity** $\dot{H} \leq \left(\frac{\partial H}{\partial p} \right)^\top G(q)u$
- **Full actuation** $m = n, G(q) = I$ we can assign **any** function of q with $\hat{u}(q) = -\frac{\partial H_a}{\partial q}(q)$.
- **Asymptotic** stability with $v = -K_{di}\dot{q}, K_{di} > 0$
- **Underactuated case** cannot solve

$$G(q)\hat{u}(q) = -\frac{\partial H_a}{\partial q}(q)$$

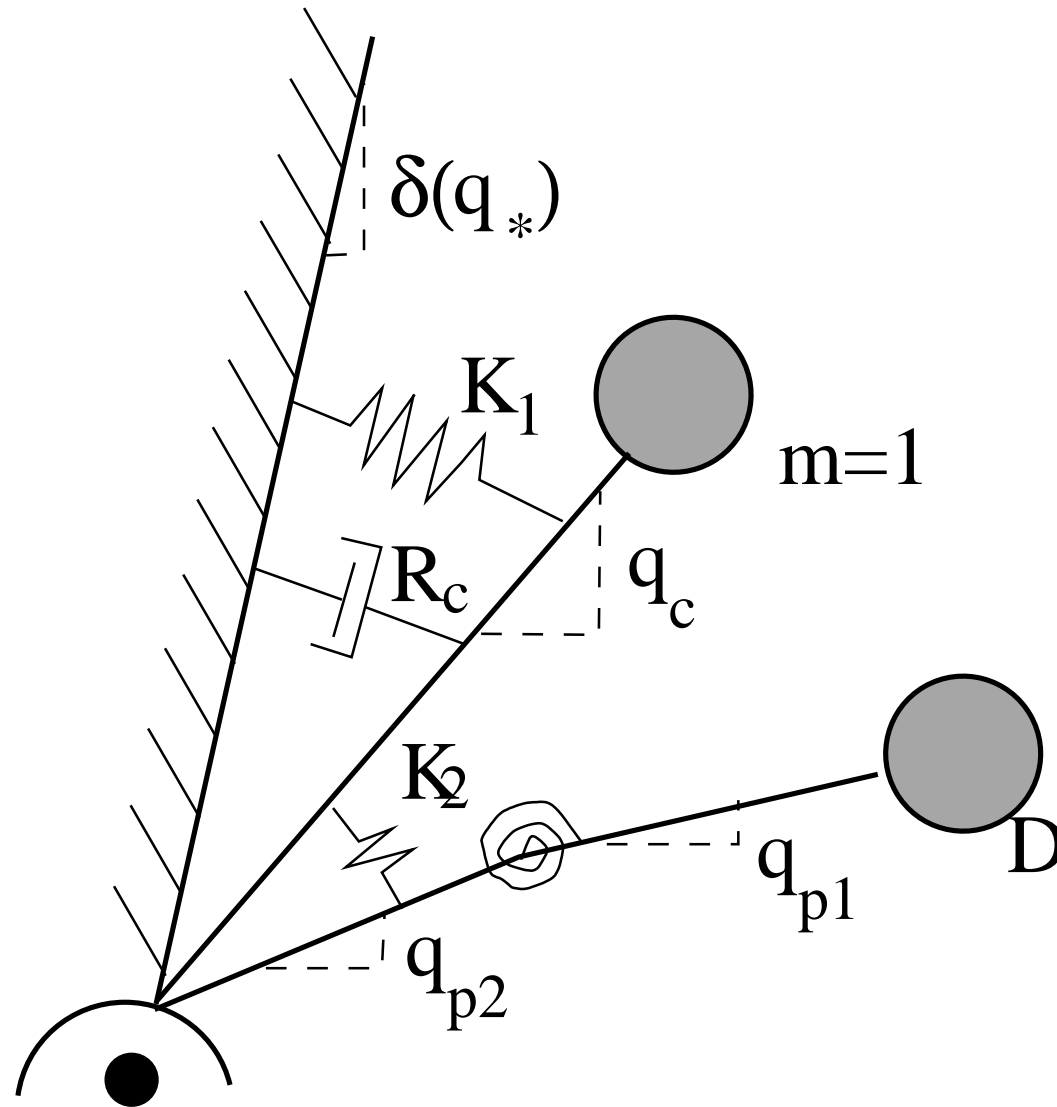
equivalent to the matching equation.

Need **total energy shaping**.

- In (Ailon/Ortega, '93) done with dynamic extension to inject damping without measuring speeds.



Flexible Pendulum



Basic IDA–PBC

Proposition Let

$$d_d = -\nabla H_d^\top F \nabla H_d$$

and denote $\hat{u} = \hat{u}_{\text{BIDA}}$.

● All solutions of the PDE

$$g^\perp F \nabla H_a = 0 \quad (\text{BIDA} - \text{PDE})$$

define assignable energy functions with

$$\hat{u}_{\text{BIDA}} = (g^\top g)^{-1} g^\top F \nabla H_a.$$

● If $\mathcal{R} \nabla H_a = 0$ and $v = 0$ then

$$\dot{H}_a = -y^\top \hat{u}_{\text{BIDA}}.$$

Remark The closed-loop system is $\dot{x} = F \nabla H_d + gv$.

Proposition Let

$$d_d = -\nabla H_d^\top F_d \nabla H_d,$$

with $F_d + F_d^\top \leq 0$, and denote $\hat{u} = \hat{u}_{\text{IDA}}$.

- All solutions of the PDE

$$g^\perp F_d \nabla H_a = g^\perp (F - F_d) \nabla H \quad (\text{IDA} - \text{PDE})$$

define assignable energy functions with

$$\hat{u}_{\text{IDA}} = (g^\top g)^{-1} g^\top [F_d \nabla H_a + (F_d - F) \nabla H].$$

- If $\mathcal{R} = \mathcal{R}_d =: -\frac{1}{2}(F_d + F_d^\top)$, $\mathcal{R} \nabla H_a = 0$ and $v = 0$ then $\dot{H}_a = -y^\top \hat{u}_{\text{IDA}}$.

Remarks

- The closed-loop is $\dot{x} = F_d \nabla H_d + gv$, hence the name IDA.
- Clearly **EBC** \Rightarrow **BIDA-PBC** \Rightarrow **IDA-PBC** but the converses are not true.

Power Shaping PBC

In PS–PBC the solution of (IDA–PDE)

$$g^\perp F_d \nabla H_a = g^\perp (F - F_d) \nabla H$$

is split in two parts. Note that, using $H_d = H + H_a$, the latter is equivalent to

$$g^\perp F_d \nabla H_d = g^\perp F \nabla H \quad (\heartsuit)$$

First, we solve (PO–PDE) $F \nabla H = F_d \nabla H_{\text{PS}}$, which replaced in (\heartsuit) yields

$$g^\perp F_d \nabla \tilde{H}_a = 0 \quad (\text{PS} - \text{PDE})$$

where we defined $\tilde{H}_a \triangleq H_d - H_{\text{PS}}$.

Remarks

- (IDA–PDE) may have solutions even though $F_d^{-1} F \nabla H$ is not a gradient of some function—as required by (PO–PDE). In other words **PS–PBC** \Rightarrow **IDA–PBC** but the converse is not true.
- PS–PBC originated, and is a natural option, for electrical circuits. See [\(PhD Jeltsema'05\)](#).

cont'd

Proposition Denote $\hat{u} = \hat{u}_{\text{PS}}$ and consider the solutions F_d , with $F_d + F_d^\top \leq 0$, of

$$\nabla \left(F_d^{-1} F \nabla H \right) = \left[\nabla \left(F_d^{-1} F \nabla H \right) \right]^\top. \quad (PO - PDE)$$

Let

$$d_d = -(F \nabla H + g \hat{u}_{\text{PS}})^\top F_d^{-1} (F \nabla H + g \hat{u}_{\text{PS}})$$

All solutions of the PDE

$$g^\perp F_d \nabla \tilde{H}_\alpha = 0 \quad (PS - PDE)$$

define assignable energy functions with

$$\hat{u}_{\text{PS}} = (g^\top g)^{-1} g^\top F_d \nabla \tilde{H}_\alpha.$$

5. Comparison of Cbl and Standard PBC: Applicability

- (CbI)
$$\begin{bmatrix} F \\ g^\top \end{bmatrix} \nabla \mathcal{C} = \begin{bmatrix} -g \\ 0 \end{bmatrix}$$

- (CbISM)
$$\begin{bmatrix} g^\perp F \\ g^\top \end{bmatrix} \nabla \mathcal{C} = 0$$

- (Basic CbI_{PS}) $F \nabla \mathcal{C} = -g$

- (CbI_{PS}) $F_d \nabla \mathcal{C} = -g$ plus (PO-PDE)
($F \nabla H = F_d \nabla H_{PS}$)

- (Basic CbI_{PS}SM)

$$g^\perp F \nabla \mathcal{C} = 0$$

- (CbI_{PS}SM)

$$g^\perp F_d \nabla \mathcal{C} = 0$$

plus (PO-PDE).

- (EBC)
$$\begin{bmatrix} g^\perp F \\ g^\top \end{bmatrix} \nabla H_\alpha = 0$$

- (Basic IDA)

$$g^\perp F \nabla H_\alpha = 0$$

- (PS)

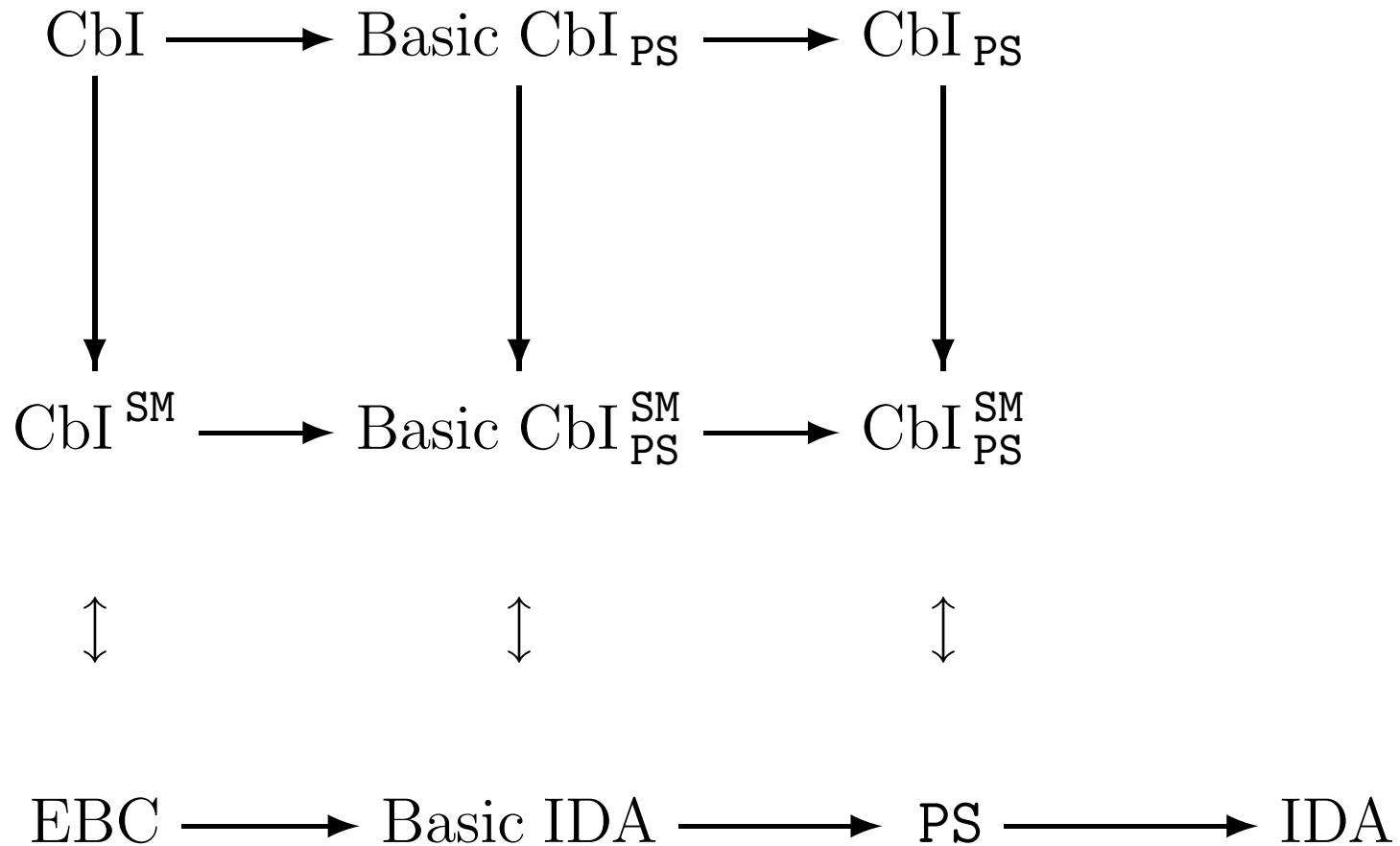
$$g^\perp F_d \nabla H_\alpha = 0$$

plus (PO-PDE)

- (IDA)

$$g^\perp F_d \nabla H_\alpha = g^\perp (F - F_d) \nabla H$$

Final Implication Diagram



Standard PBC and Cbl: Connections

Cbl

- Dynamic feedback control $u = -y_c + v = -\nabla_{\zeta} H_c(\zeta) + v$,
- ζ controllers state with energy $H_c(\zeta)$ free,
- Generate Casimir functions, \mathcal{C} , that make $\Omega = \{(x, \zeta) | \zeta = \mathcal{C}(x)\}$ invariant
- \Rightarrow For arbitrary Φ

$$\dot{H}(x) + \dot{H}_c(\zeta) + \Phi(\mathcal{C}(x) - \zeta) \leq v^{\top} y$$

Standard PBC

- Solve some PDE on H_a and define a static state feedback, $\hat{u}(x)$, that ensures

$$\dot{H} + \dot{H}_a \leq v^{\top} y$$

Questions

- Is there a connection between the two methods?
- What happens if we restrict to Ω ?
- Is there an advantage of dynamic extension from minimum assignment viewpoint?

Restricting a Cbl Controller Yields an EBC

Proposition Assume (Cbl–PDE) admit a solution. Then, for all $H_c : \mathbb{R}^m \rightarrow \mathbb{R}$, the PH system $\Sigma_{(u,y)}$ in closed–loop with the **static state–feedback** control $u = \hat{u}_{\text{EB}}(x) + v$, where

$$\hat{u}_{\text{EB}}(x) = -\nabla_{\mathcal{C}} H_c(\mathcal{C}(x)),$$

satisfies the cyclo–passivity inequality

$$\dot{H} + \frac{d}{dt} H_c(\mathcal{C}(x)) \leq v^\top y.$$

Furthermore,

$$\frac{d}{dt} H_c(\mathcal{C}(x)) = -y^\top \hat{u}_{\text{EB}}.$$

Proof Define $H_a(x) \triangleq H_c(\mathcal{C}(x))$

$$\begin{aligned} \dot{H}_a &= (\nabla_{\mathcal{C}} H_c(\mathcal{C}))^\top (\nabla \mathcal{C})^\top (F \nabla H + g u) \\ &= (\nabla_{\mathcal{C}} H_c(\mathcal{C}))^\top g^\top \nabla H \quad (\Leftarrow F^\top \nabla \mathcal{C} = g, g^\top \nabla \mathcal{C} = 0) \\ &= -\hat{u}_{\text{EB}}^\top y \quad (\Leftarrow \hat{u}_{\text{EB}} = -\nabla_{\mathcal{C}} H_c(\mathcal{C}), y = g^\top \nabla H). \end{aligned}$$

Restricting a CbI_{PS} Controller Yields an IDA–PBC

Proposition Assume the conditions for CbI_{PS} are satisfied. Then, for all $H_c : \mathbb{R}^m \rightarrow \mathbb{R}$, the state–feedback controller

$$\hat{u}_{IDA}(x) = -\nabla_{\mathcal{C}} H_c(\mathcal{C}(x)),$$

ensures that the IDA–PBC matching condition

$$F\nabla H + g\hat{u}_{IDA} = F_d\nabla H_d$$

is satisfied with $H_d = H_{PS} + H_a$ and $H_a(x) \triangleq H_c(\mathcal{C}(x))$.

Proof

Conditions for CbI_{PS} :

- (PO–PDE) $\Leftrightarrow F\nabla H = F_d\nabla H_{PS}$,
- (CbI_{PS} –PDE) $\Leftrightarrow F_d\nabla \mathcal{C} = -g$.

Replacing in the matching equation yields

$$F_d(\nabla H_{PS} - (\nabla \mathcal{C})\hat{u}_{IDA}) = F_d\nabla H_d \quad \Leftrightarrow \quad \nabla H_a = -(\nabla \mathcal{C})\hat{u}_{IDA},$$

which is satisfied with H_a and \hat{u}_{IDA} above. ◇ ◇ ◇

Dynamic Extension and Stabilization

- We have concentrated our attention on the ability of the various PBCs to modify the energy function, without particular concern to stabilization.
- Stability will be ensured if a (desired) strict minimum is assigned to the total energy function

Proposition In the single input case, the use of a dynamic extension does not provide any additional freedom for minimum assignment to the corresponding static state–feedback solutions.

Proof Define

$$\begin{aligned} W(x, \zeta) &\triangleq H(x) + H_c(\zeta) + \Phi(\mathcal{C}(x) - \zeta) \\ H_d(x) &\triangleq H(x) + H_c(\mathcal{C}(x)). \end{aligned}$$

We will prove that

$$\nabla W_{\star} = 0 \quad \text{and} \quad \nabla^2 W_{\star} > 0 \quad \Rightarrow \quad (\nabla H_d)_{\star} = 0 \quad \text{and} \quad (\nabla^2 H_d)_{\star} > 0.$$

cont'd

Compute

$$\nabla W = \begin{bmatrix} \nabla H + \Phi' \nabla \mathcal{C} \\ H'_c - \Phi' \end{bmatrix}, \quad \nabla H_d = \nabla H + H'_c \nabla \mathcal{C}.$$

Now,

$$\nabla W_\star := \nabla W(x_\star, \zeta_\star) = 0 \Rightarrow \Phi'_\star = (H'_c)_\star \Rightarrow \nabla_x W_\star = (\nabla H_d)_\star.$$

On the other hand,

$$\begin{aligned} \nabla^2 H_d &= \nabla^2 H + H'_c \nabla^2 \mathcal{C} + H''_c \nabla \mathcal{C} \nabla \mathcal{C}^\top \\ \nabla^2 W &= \begin{bmatrix} \nabla^2 H + \Phi' \nabla^2 \mathcal{C} & 0 \\ 0 & H''_c \end{bmatrix} + \Phi'' \begin{bmatrix} \nabla \mathcal{C} \nabla \mathcal{C}^\top & -\nabla \mathcal{C} \\ -\nabla \mathcal{C}^\top & 1 \end{bmatrix} \end{aligned}$$

cont'd

Now,

$$\begin{bmatrix} I & \nabla \mathcal{C} \\ 0 & |\nabla \mathcal{C}|^2 \end{bmatrix} \nabla^2 W \begin{bmatrix} I & \nabla \mathcal{C} \\ 0 & |\nabla \mathcal{C}|^2 \end{bmatrix}^\top = \begin{bmatrix} \nabla^2 H + \Phi' \nabla^2 \mathcal{C} + H_c'' \nabla \mathcal{C} \nabla \mathcal{C}^\top & * \\ * & * \end{bmatrix}$$

From Sylvester's Law of Inertia we have

$$\begin{aligned} \nabla^2 W_\star > 0 &\Rightarrow (\nabla^2 H + \Phi' \nabla^2 \mathcal{C} + H_c'' \nabla \mathcal{C} \nabla \mathcal{C}^\top)_\star > 0 \\ &\Leftrightarrow (\nabla^2 H_d)_\star > 0 \quad (\Leftarrow \Phi'_\star = (H'_c)_\star) \end{aligned}$$

◇◇◇

Future Research

- Is there a Cbl version of IDA? What is the modification that is needed to add this degree of freedom?
- CbI_{PS}^{SM} (or PS) suggest a procedure to solve the non-homogenous PDE of IDA

$$g^\perp F_d \nabla H_d = g^\perp F \nabla H.$$

Instead of fixing F_d and solving the PDE for H_d , CbI_{PS}^{SM} splits the equation into two homogeneous PDEs:

$$\nabla \left(F_d^{-1} F \nabla H \right) = \left[\nabla \left(F_d^{-1} F \nabla H \right) \right]^\top, \quad g^\perp F_d \nabla H_a = 0.$$

It is interesting to explore this decomposition as an alternative for generation of F_d . The parametrization of the solutions given above is of particular importance.

- We have fixed the order of the dynamic extension to be m . There are some advantages for increasing their number. Also, we have taken simple nonlinear integrators.
- Dynamic extension does not help for minimum assignment, but certainly has an impact on performance and simplicity.