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Lectures on Optimal Control :
Optimal Control on Lie Groups: Integrable Hamiltonian systems

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Abstract

Variational problems with constraints are most naturally phrased as optimal control problems in whcih case the Maximum Principle with its associated Hamiltonian formalism leads efficiently to their solutions. The aim of these lectures is to demonstrate the advantages of this point of view by focusing on problems with Lie group symmetries. Typically we will concentrate on the following class of problems:

Minimize the integral $\int_0^T < u(t), u(t) > dt$ over the solutions $x(t)$ that satisfy given boundary conditions of a control system in a Lie group G

$$\frac{dx}{dt} = X_0(x(t)) + \sum_{i=1}^{m} u_i(t) X_i(x(t)) \tag{1a}$$

where $X_0, \ldots, X_m$ are either left or right invariant vector fields on G and where $< \, , \, >$ denotes a positive definite quadratic form in the control space $u = (u_1, \ldots, u_m)$.

The existence of the optimal solutions is intimately connected to the questions of controllability which will be discussed briefly at the beginning of the lectures. Then we will move on to the Hamiltonian systems induced by the Maximum Principle on the cotangent bundle T^*G realized as the product $G \times \mathfrak{g}^*$, where $\mathfrak{g}^*$ dentoes the dual of the Lie algebra $\mathfrak{g}$ of G.This trivialization of the cotangent bundle is done by either left or right translations dictated by the symmetries of (1a). As a further consequence of the symmetries of the above optimal problem, the Hamiltonians H induced by the Maximum Principle are functions on $\mathfrak{g}$. The associated Hamiltonian flows $\vec{H}$ then admit a triangular form

$$\frac{dx}{dt} = x(t)dH(l(t)), \frac{dl}{dt} = -ad^*dH(l(t)(l(t), (x(t), l(t)) \in G \times \mathfrak{g}^*. \tag{2}$$

in which the projected system $\frac{dl}{dt} = -ad^*dH(l(t)(l(t)$.on $\mathfrak{g}^*$ plays a crucial part.

The recognition of $\mathfrak{g}^*$ as a Poisson manifold, foliated by the coadjoint orbits of G as the symplectic leaves, forms a natural theoretical foundation for understanding the solutions of such systems. This aspect of the theory will be explained in considerable detail with a particular emphasis on various notions of integrability, i.e., on the solvability of the Hamiltonian equations in terms of Abelian integrals.

The relevance of this theory to problems of mechanics and geometry will be illustrated through mechanical tops. and their relations to the elastic problems

of Kirchhoff, geodesic problems of C.L Jacobi and J. Moser and the gravitational problem of Kepler.

Tentative outline of the lectures:

Lecture 1. The orbit theorem and its applications to controllability

Lecture 2. Hamiltonian systems on Lie groups. Poisson manifolds and coadjoint orbits.

Lecture 3. Symmetric spaces and the associated Hamiltonian systems.

Lecture 4. Elastic problems and their relations to mechanical tops..

Lecture 5. Lax pairs, complete integrability, Jacobi, Kepler and Moser.

References

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