Discrete Mechanics and Groupoids

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Outline of the Talk

- Why is it important Geometric integration?
- Discrete Mechanics
- Groupoids and Mechanics

Why is it important Geometric Integration?

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Words written at the entrance of the Plato's Temple

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- Motion is described by differential equations derived from laws of physics

$$\frac{d^2x}{dt^2} = F(t, x, \frac{dx}{dt})$$

- The equations contains not just a statement of acceleration but *all the physical laws relevant* (phase space, symmetries, invariance properties...)

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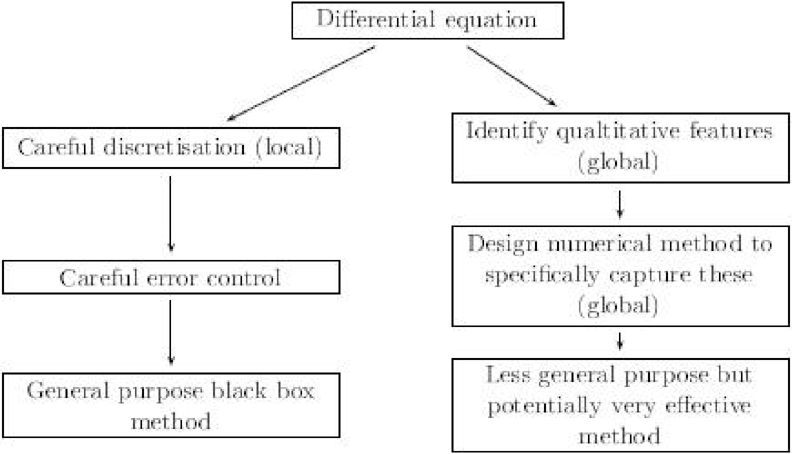
Differential equation
Careful discretisation (local)
Careful error control
General purpose black box method
Identify qualtitative features (global)
Design numerical method to specifically capture these (global)
Less general purpose but potentially very effective method

- **Conservation laws**. Functions that stay constant along the solution trajectories. For example, the energy $H(q(t), p(t))$ of the Hamiltonian system

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}(q,p)\,, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}(q,p)$$

  remains constant along a solution trajectory.
  $p, q \in \mathbb{R}^d \Longrightarrow \mathcal{M} = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \mid H(x,y) = H(q(0), p(0))\}$
  Differentiable manifolds

- **Symmetries**. Transformations which, when applied to dependent or independent variables, gives another solution to the same system of differential equations.
  Lie groups

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- **Symplectic structure** in Hamiltonian systems.
  Symplecticity:

$$\frac{\partial(p(t), q(t))}{\partial(p(0), q(0))}^{\top} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \frac{\partial(p(t), q(t))}{\partial(p(0), q(0))} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, t \geqslant 0$$

- Standard methods for simulating motion called numerical integrators completely ignore all of the previous hidden physical laws.
- Since about 1990 new methods have been developed called **geometric integrators** which obey some of these extra laws.

Disadvantages:

1. The hidden physical law usually has to be known if the integrator is going to obey it.
2. Because we are asking something more of our method it may turn out to be computationally more expensive than a standard method.
3. Many systems have multiple hidden laws for which methods are currently known which preserve any one law but not all simultaneously.

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Advantages

1. Simulations can be run for enormously long times because there are no spurious non physical effecs such as dissipation of energy in a conservative systems.
2. By studying the structure of the equations very simple fast and reliable geometric integrators can often be found.
3. In some situations results can be guaranteed to be qualitatively correct.
4. For some systems even the actual quantitative errors are much smaller for short, medium and long times than in standard methods.

It is natural to look forward to those discrete systems which preserve as much as possible the intrinsic properties of the continuous system.

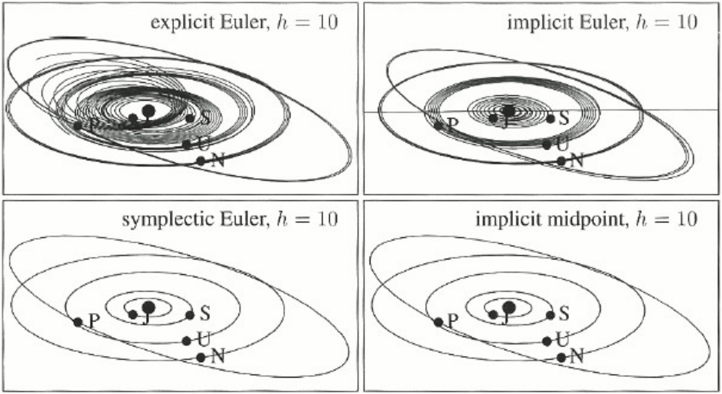
Feng Kang 1985

Figure: (Feng Kang 1920-1993)

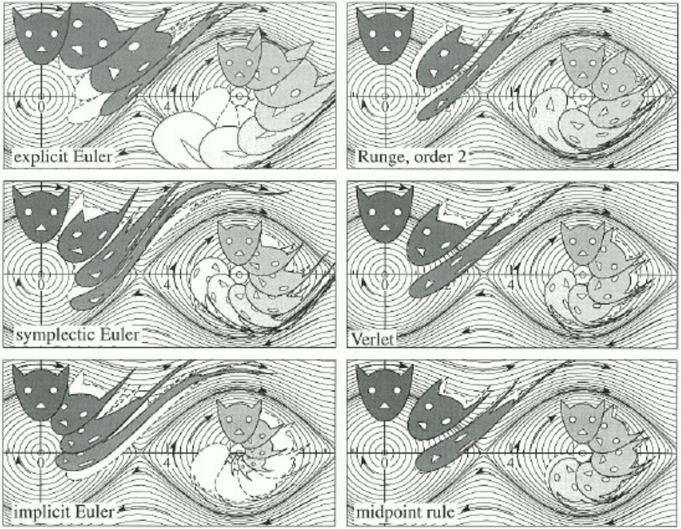
Differential geometry (Geometric mechanics)

$$\Downarrow$$

Geometric integration



(Hairer, Lubich and Wanner 2002: Geometric Numerical Integration, page 12)



(Hairer, Lubich Wanner 2002: Geometric Numerical Integration, page 176)

They throw geometry out of the door, and it comes back through the window.

H.G. Folder, Auckland 1973,
reading new mathematics at the age of 84

> Obtain numerical integrators from a discrete version of variational Hamilton's principle

Define the set $C^2(q_0, q_1; [t_0, t_1])$ as the C^2-curves $\sigma : [t_0, t_1] \to Q$ such $q(t_0) = q_0$ and $q(t_1) = q_1$.

$$\begin{array}{rccc} F : & C^2(q_0, q_1; [t_0, t_1]) & \longrightarrow & \mathbb{R} \\ & q(\cdot) & \longmapsto & \int_{t_1}^{t_0} L(t, q(t), \dot{q}(t))\, dt \end{array}$$

L lagrangian function $L : \mathbb{R} \times TQ \to \mathbb{R}$.

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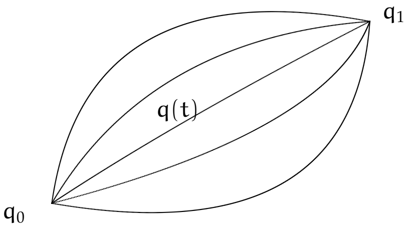

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Hamilton's principle

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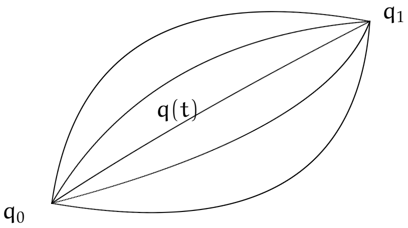

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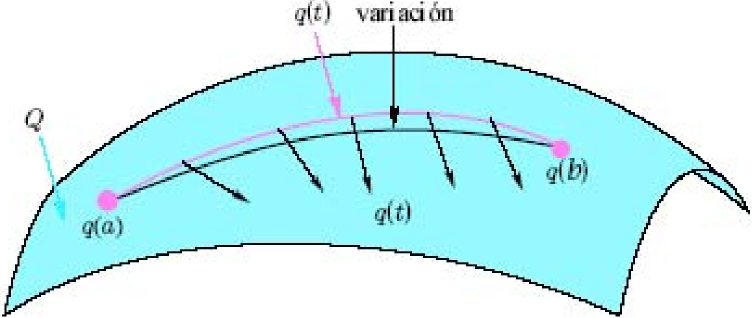


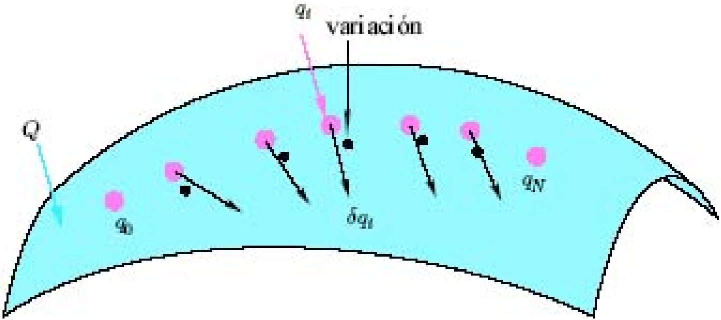
Hamilton's principle $\Longrightarrow \delta \int_{t_0}^{t_1} L(t, q(t), \dot{q}(t))\, dt = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0,\ 1 \leqslant i \leqslant n = \dim Q$$

$$L_d(q_0, q_1, t_0, t_1) \approx \int_{t_0}^{t_1} L(t, q(t), \dot{q}(t))\, dt$$

$$L_d(q_0, q_1, h) \approx \int_0^h L(t, q(t), \dot{q}(t))\, dt$$

For h fixed, $L_d : Q \times Q \longrightarrow \mathbb{R}$



Q n-dimensional differentiable manifold $L_d : Q \times Q \longrightarrow \mathbb{R}$ discrete lagrangian.

Action: $S : Q^{N+1} \longrightarrow \mathbb{R}$

$$S = \sum_{k=0}^{N-1} L_d(q_k, q_{k+1})$$

Discrete Euler-Lagrange equations (DEL))

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = 0$$

$$\Downarrow$$

$$\begin{array}{rccc} F_{L_d} : & Q \times Q & \longrightarrow & Q \times Q \\ & (q_0, q_1) & \longmapsto & (q_1, q_2) \end{array}$$

Discrete flow

Momentum preservation

L_d is G-invariant

$$J_d : Q \times Q \longrightarrow \mathfrak{g}^*$$

$$\langle J_d(x, y), \xi \rangle = \langle D_2 L_d(x, y), \xi_Q(y) \rangle$$

$$\langle D_2 L_d(q_{k-1}, q_k), \xi_Q(q_k) \rangle = \langle D_2 L_d(q_k, q_{k+1}), \xi_Q(q_{k+1}) \rangle$$

Symplecticity

$$\begin{array}{rcll} \mathbb{F}^{-}L_d: & Q\times Q & \longrightarrow & T^*Q \\ & (x,y) & \longmapsto & (x,-D_1L_d(x,y)) \end{array}$$

$$\begin{array}{rcll} \mathbb{F}^{+}L_d: & Q\times Q & \longrightarrow & T^*Q \\ & (x,y) & \longmapsto & (y,D_2L_d(x,y)) \end{array}$$

$$\omega_{L_d} = (\mathbb{F}^{\pm}L_d)^*\omega_Q$$

The discrete flow preserves the symplectic form $F^*_{L_d}(\omega_{L_d}) = \omega_{L_d}$

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Discrete momentum

Denote by

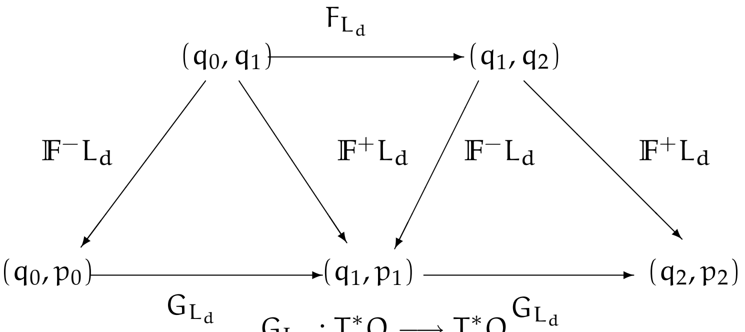
$$\begin{aligned} p^+_{k,k+1} &= p^+(q_k, q_{k+1}) = \mathbb{F}^+L_d(q_k, q_{k+1}) \\ p^-_{k,k+1} &= p^-(q_k, q_{k+1}) = \mathbb{F}^-L_d(q_k, q_{k+1}) \end{aligned}$$

$$\begin{gathered} \text{Discrete Euler-Lagrange equations} \\ \Updownarrow \\ p^+_{k-1,k} = p^-_{k,k+1} \\ \Updownarrow \\ \mathbb{F}^+L_d(q_{k-1}, q_{k+1}) = \mathbb{F}^-L_d(q_k, q_{k+1}) \end{gathered}$$

A unique momentum at time kh.

$$\mathbb{F}^+L_d = \mathbb{F}^-L_d \circ F_{L_d}$$

Discrete hamiltonian flow



$$G_{L_d} : T^*Q \longrightarrow T^*Q$$

$$\begin{aligned} G_{L_d} &= \mathbb{F}^+ L_d \circ F_{L_d} \circ (\mathbb{F}^+ L_d)^{-1} \\ G_{L_d} &= \mathbb{F}^- L_d \circ F_{L_d} \circ (\mathbb{F}^- L_d)^{-1} \\ G_{L_d} &= \mathbb{F}^+ L_d \circ (\mathbb{F}^- L_d)^{-1} \end{aligned}$$

$$\boxed{G^*_{L_d} \omega_Q = \omega_Q}$$

Discrete exact lagrangian

Let L be a regular lagrangian

$$L_d^E(q_0, q_1, h) = \int_0^h L(c(t), \dot{c}(t))\, dt$$

for q_0 and q_1 enough close and h small. Here $c : I \subseteq \mathbb{R} \to Q$ is the unique solution of the Euler-Lagrange equations for L which satisfies $c(0) = q_0$ and $c(h) = q_1$.

Lemma: Denote by $\mathbb{F}L : TQ \longrightarrow T^*Q$ the Legendre transformation, then

$$\begin{aligned} \mathbb{F}^+L_d^E(q_0, q_1, h) &= \mathbb{F}L(c(h), \dot{c}(h)) \\ \mathbb{F}^-L_d^E(q_0, q_1, h) &= \mathbb{F}L(c(0), \dot{c}(0)) \end{aligned}$$

Proof

$$\begin{aligned} \mathbb{F}^-L_d^E(q_0, q_1, h) &= -\int_0^h \left[\frac{\partial L}{\partial q} \cdot \frac{\partial c}{\partial q_0} + \frac{\partial L}{\partial \dot{q}} \cdot \frac{\partial \dot{c}}{\partial q_0}\right] dt \\ &= -\int_0^h \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right] \cdot \frac{\partial c}{\partial q_0}\, dt - \frac{\partial L}{\partial \dot{q}} \cdot \frac{\partial c}{\partial q_0}\Big|_0^h \\ &= \frac{\partial L}{\partial \dot{q}}(c(0), \dot{c}(0)) = \mathbb{F}L(c(0), \dot{c}(0)) \end{aligned}$$

If the discrete Lagrangian is an approximation to a continuous Lagrangian then the discrete system is a numerical integrator for the continous system.

- **Symplectic**
- **Momentum preserving**

E. Hairer, C. Lubic and G. Wanner: *Geometric Numerical Integration, Structure-Preserving Algorithms for Ordinary Differential Equations*, Springer Series in Computational Mathematics **31**, Springer-Verlag Berlin Heidelberg, 2002.

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Reduction of mechanical systems: Euler-Poincaré reduction.

Let G be a Lie group and let G act on itself by left translation and hence, by tangent lift, on its tangent bundle TG. Let $L : TG \to \mathbb{R}$ be a G-invariant Lagrangian.

Then, L is completely determined by its restriction to the tangent space at the identity e. Identifying T_eG with the Lie algebra $\mathfrak{g}$ of G, we define $l : \mathfrak{g} \to \mathbb{R}$.

The velocity of the system is given by $\dot{g}(t)$, thought of as a tangent vector to G at $g(t)$. The body velocity is defined by $\xi(t) = g(t)^{-1}\dot{g}(t)$, the left translation of $\dot{g}(t)$ to the identity.

Theorem (Euler-Poincaré reduction)

A curve $g(t)$ in G satisfies the Euler-Lagrange equations for L iff $\xi(t)$ satisfies the Euler-Poincaré equations for l:

$$\frac{d}{dt}\frac{\partial l}{\partial \xi} = ad^*_{\xi}\frac{\partial l}{\partial \xi}$$

Theorem

Let G be a Lie group and $L_d : G \times G \longrightarrow \mathbb{R}$ a discrete lagrangian left G-invariant ($L_d(hg_k, hg_{k+1}) = L_d(g_k, g_{k+1})$ for all $h \in G$). For a curve $(g_k, g_{k+1}) \in G \times G$, consider $W_k = g_k^{-1} g_{k+1}$ and defined the reduced lagrangian $l_d : G \longrightarrow \mathbb{R}$ by
$l_d(W_k) = L_d(e, W_k)$.

Then, the following are equivalent

(i) $(g_k)_{0\leqslant k\leqslant N}$ satisfies the Discrete Euler-lagrange equations for L_d ;

(ii) $(g_k)_{1\leqslant k\leqslant N}$ extremizes the discrete action

$$(g_k)_{0\leqslant k\leqslant N} \longmapsto \sum_{k=0}^{N} L_d(g_k, g_{k+1})$$

for variations with fixed initial and final points.

(iii) $(W_k)_{0\leqslant k\leqslant N-1}$ satisfies the following *discrete Euler-Poincaré equations*:

$$R^*_{W_k} l'_d(W_k) - L^*_{W_{k-1}} l'_d(W_{k-1}) = 0, k = 1, \ldots, N-1$$

(iv) $(W_k)_{1\leqslant k\leqslant N-1}$ extremizes $(W_k)_{0\leqslant k\leqslant N-1} \longmapsto \sum_{k=0}^{N-1} l_d(W_k)$ for variations of the form $\delta W_k = -\Sigma_k W_k + W_k \Sigma_{k+1}$ with $\Sigma_0 = \Sigma_N = 0$.

Reduction of mechanical systems: Lagrange-Poincaré reduction.

Assume Q is Riemannian (the metric often being the kinetic energy metric) and that G acts on Q freely by isometries, so $Q \to Q/G$ is a principal bundle. If we declare the horizontal spaces to be metric orthogonal to the group orbits, this uniquely defines a connection called the mechanical connection. The space Q/G is called shape space.

$TQ/G \cong T(Q/G) \oplus \tilde{\mathfrak{g}}$

If we have an invariant Lagrangian on TQ it induces a Lagrangian $\mathfrak{l}$ on $(TQ)/G$ and hence on $T(Q/G) \oplus \tilde{\mathfrak{g}}$.

Reduced Euler-Lagrange equations

$$L : TQ \longrightarrow \mathbb{R}$$

$$L : \mathfrak{g} \longrightarrow \mathbb{R}$$

$$L : (TQ)/G \longrightarrow \mathbb{R}$$

IS IT POSSIBLE TO CONSTRUCT VARIATIONAL INTEGRATORS FOR REDUCED SYSTEMS?

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The pair or banal groupoid

$$Q \times Q$$

$$\begin{array}{lcl}
\alpha : Q \times Q \longrightarrow Q & ; & (q, q') \longrightarrow q, \\
\beta : Q \times Q \longrightarrow Q & ; & (q, q') \longrightarrow q', \\
\epsilon : Q \longrightarrow Q \times Q & ; & q \longrightarrow (q, q), \\
m : (Q \times Q)_2 \longrightarrow Q \times Q & ; & (q, q'), (q', q'') \longrightarrow (q, q''), \\
i : Q \times Q \longrightarrow Q \times Q & ; & (q, q') \longrightarrow (q', q).
\end{array}$$

The product manifold $Q \times Q \rightrightarrows Q$ is a Lie groupoid over Q.

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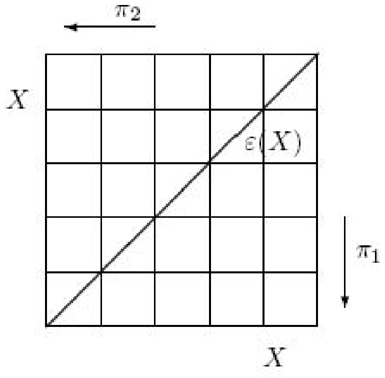


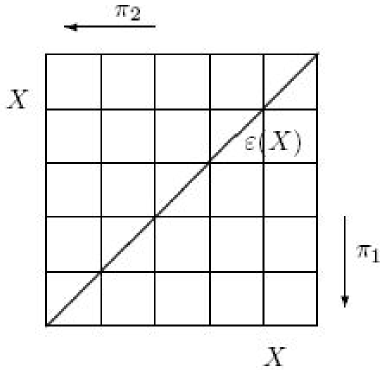
If q is a point of Q, it follows that

$$V_{\epsilon(q)}\alpha = \{0_q\} \times T_q Q \subseteq T_q Q \times T_q Q \cong T_{(q,q)}(Q \times Q).$$

Thus, the linear maps

$$\Psi_q : T_q Q \to V_{\epsilon(q)}\alpha, \quad v_q \to (0_q, v_q),$$

induce an isomorphism (over the identity of Q).



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- **Groupoids** were introduced by **H. Brandt** in 1926:

  H. Brandt: Über eine Verallgemeinerung des Gruppenbegriffes, Math. Ann. **96** *(1926), 360-366.*

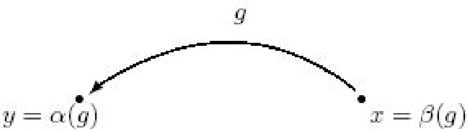
  Brandt's definition of groupoid arose out of his work on generalizing to quaternary quadratic forms a composition of binary quadratic forms due to Gauss.
- The use of groupoid was expanded greatly by **C. Ehresmann** from 1950 in various main areas (differential geometry and differential topology). Groupoid techniques in foliation theory were developped by **Haefliger**.

- In algebraic geometry, **A. Grothendiek** used groupoids extensively, especially, to study the equivalence relations which arise in the construction of moduli spaces.
- **G. Mackey** in analysis, **A. Connes** in noncommutative geometry, **P. Higgins**, **R. Brown** in algebraic topology, etc
- **J. Pradines** for the extension of Lie theory from differentiable groups to groupoids. See also the book by **K. Mackenzie**...
- **A. Weinstein** for the extension to MECHANICS...

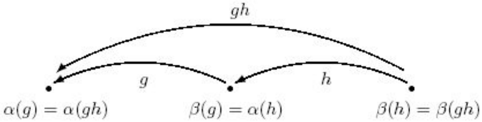
A **groupoid** over a set M is a set G together with the following structure maps:

- A pair of maps $\alpha : G \to M$, the **source**, and $\beta : G \to M$, the **target**. These maps define the set of composable pairs

$$G_2 = \{(g, h) \in G \times G / \beta(g) = \alpha(h)\}.$$

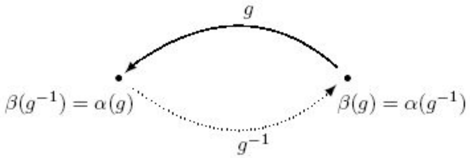


- A **multiplication** $m : G_2 \to G$, to be denoted simply by $m(g,h) = gh$, such that
  - $\alpha(gh) = \alpha(g)$ and $\beta(gh) = \beta(h)$.
  - $g(hk) = (gh)k$.

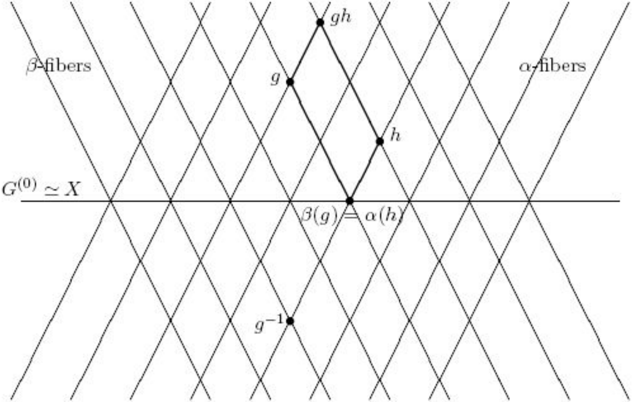


- An **identity section** $\epsilon : M \to G$ such that
  - $\epsilon(\alpha(g))g = g$ and $g\epsilon(\beta(g)) = g$.

- An **inversion map** $i : G \to G$, to be denoted simply by $i(g) = g^{-1}$, such that
  - $g^{-1}g = \epsilon(\beta(g))$ and $gg^{-1} = \epsilon(\alpha(g))$.



The groupoid $G \rightrightarrows M$ is said to be a **Lie groupoid** if G and M are manifolds and all the structure maps are differentiable with α and β differentiable submersions.


gh
β-fibers
g
α-fibers
h
$G^{(0)} \simeq X$
$\beta(g) = \alpha(h)$
g^{-1}

If $x \in M$, $\alpha^{-1}(x)$ (resp., $\beta^{-1}(x)$) will be said the *α-fiber* (resp., the *β-fiber*) of x.

If $g \in G$ then the **left-translation** by $g \in G$ and the **right-translation** by g are the diffeomorphisms

$$\begin{array}{lll} l_g : \alpha^{-1}(\beta(g)) \longrightarrow \alpha^{-1}(\alpha(g)), & ; & h \longrightarrow l_g(h) = gh, \\ r_g : \beta^{-1}(\alpha(g)) \longrightarrow \beta^{-1}(\beta(g)), & ; & h \longrightarrow r_g(h) = hg. \end{array}$$

A vector field $\tilde{X}$ on G is said to be **left-invariant** (resp., **right-invariant**) if it is tangent to the fibers of α (resp., β) and $\tilde{X}(gh) = (T_h l_g)(\tilde{X}_h)$ (resp., $\tilde{X}(gh) = (T_g r_h)(\tilde{X}(g))$, for $(g, h) \in G_2$.

Lie algebroid associated with G

We consider the vector bundle $\tau : AG \to M$, whose fiber at a point $x \in M$ is

$$A_x G = V_{\epsilon(x)}\alpha = \mathrm{Ker}(T_{\epsilon(x)}\alpha)$$

It is easy to prove that there exists a bijection between the space $\Gamma(\tau)$ an the set of left-invariant (resp., right-invariant) vector fields on G. If X is a section of $\tau : AG \to M$, the corresponding left-invariant (resp., right-invariant) vector field on G will be denoted $\overleftarrow{X}$ (resp., $\overrightarrow{X}$).

$$\overleftarrow{X}(g) = (T_{\epsilon(\beta(g))} l_g)(X(\beta(g))),$$

$$\overrightarrow{X}(g) = (T_{\epsilon(\alpha(g))} r_g)\{X(\alpha(g)) - (T_{\alpha(g)}\epsilon)((T_{\epsilon(\alpha(g))}\beta)(X(\alpha(g))))\},$$

Using the above facts, we may introduce a Lie algebroid structure on AG:

$$\overleftarrow{[\![X, Y]\!]} = [\overleftarrow{X}, \overleftarrow{Y}], \quad \rho(X)(x) = (T_{\epsilon(x)}\beta)(X(x)),$$

for $X, Y \in \Gamma(\tau)$ and $x \in M$.
Note that

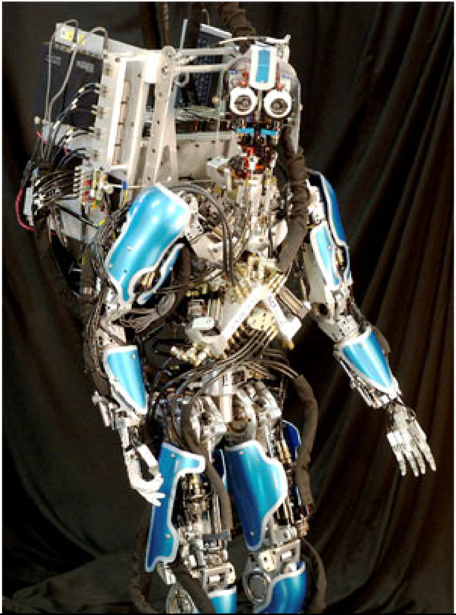
$$\overrightarrow{[\![X, Y]\!]} = -[\overrightarrow{X}, \overrightarrow{Y}], \quad [\overrightarrow{X}, \overleftarrow{Y}] = 0,$$

Given a section X of $\tau : AG \longrightarrow M$, then the corresponding left and right-invariant representatives are related by the inversion map:

$$Ti \circ \overleftarrow{X} = -\overrightarrow{X} \circ i \qquad \text{and} \qquad Ti \circ \overrightarrow{X} = -\overleftarrow{X} \circ i.$$

BUT................

ANDROIDS



Examples of Lie groupoids

1. **Lie groups** G.

$$\alpha(g) = e, \quad \beta(g) = e, \quad \epsilon(e) = e, \quad i(g) = g^{-1}, \quad m(g,h) = gh$$

$$G \rightsquigarrow \mathfrak{g}$$

2. **The banal groupoid** $Q \times Q$.

$$\alpha(x,y) = x, \quad \beta(x,y) = y, \quad \epsilon(x) = (x,x), \quad i(x,y) = (y,x)$$
$$m((x,y),(y,z)) = (x,z)$$

$$Q \times Q \rightsquigarrow TQ$$

Action Lie groupoids

Let $G \rightrightarrows M$ be a Lie groupoid and $\pi : P \to M$ be a smooth map. If $P\ {}_\pi\!\times_\alpha G = \{(p,g) \in P \times G / \pi(p) = \alpha(g)\}$ then a right action of G on π is a smooth map

$$P\ {}_\pi\!\times_\alpha G \to P, \quad (p,g) \to pg,$$

which satisfies the following relations

$$\begin{array}{rcll} \pi(pg) & = & \beta(g), & \text{for } (p,g) \in P\ {}_\pi\!\times_\alpha G, \\ (pg)h & = & p(gh), & \text{for } (p,g) \in P\ {}_\pi\!\times_\alpha G \text{ and } (g,h) \in G_2, \text{ and} \\ p\epsilon(\pi(p)) & = & p, & \text{for } p \in P. \end{array}$$

Given such an action one constructs *the action Lie groupoid* $P\ {}_\pi\!\times_\alpha G$ over P by defining

$$\begin{array}{lcl} \tilde{\alpha}_\pi : P\ {}_\pi\!\times_\alpha G \longrightarrow P & ; & (p,g) \longrightarrow p, \\ \tilde{\beta}_\pi : P\ {}_\pi\!\times_\alpha G \longrightarrow P & ; & (p,g) \longrightarrow pg, \\ \tilde{\epsilon}_\pi : P \longrightarrow P\ {}_\pi\!\times_\alpha G & ; & p \longrightarrow (p, \epsilon(\pi(p))), \\ \tilde{m}_\pi : (P\ {}_\pi\!\times_\alpha G)_2 \longrightarrow P\ {}_\pi\!\times_\alpha G & ; & ((p,g),(pg,h)) \longrightarrow (p,gh), \\ \tilde{i}_\pi : P\ {}_\pi\!\times_\alpha G \longrightarrow P\ {}_\pi\!\times_\alpha G & ; & (p,g) \longrightarrow (pg, g^{-1}). \end{array}$$

Now, if $p \in P$, we consider the map $p\,\cdot : \alpha^{-1}(\pi(p)) \to P$ given by

$$p \cdot (g) = pg.$$

Then, if $\tau : AG \to M$ is the Lie algebroid of G, the $\mathbb{R}$-linear map $\Phi : \Gamma(\tau) \to \mathfrak{X}(P)$ defined by

$$\Phi(X)(p) = (T_{\epsilon(\pi(p))} p\,\cdot)(X(\pi(p))), \quad \text{for } X \in \Gamma(\tau) \text{ and } p \in P,$$

induces an action of AG on $\pi : P \to M$. In addition, the Lie algebroid associated with the Lie groupoid $P\; {}_{\pi}\!\times_{\alpha} G \rightrightarrows P$ is the action Lie algebroid $AG \ltimes \pi$.

3 **Atiyah or gauge groupoids**. Let $p : Q \to M$ be a principal G-bundle. Then, the free action, $\Phi : G \times Q \to Q$, $(g, q) \to \Phi(g, q) = gq$, of G on Q induces, in a natural way, a free action $\Phi \times \Phi : G \times (Q \times Q) \to Q \times Q$ of G on $Q \times Q$ given by $(\Phi \times \Phi)(g, (q, q')) = (gq, gq')$, for $g \in G$ and $(q, q') \in Q \times Q$. Moreover, one may consider the quotient manifold $(Q \times Q)/G$ and it admits a Lie groupoid structure over M with structural maps given by

$$
\begin{array}{lcl}
\tilde{\alpha} : (Q \times Q)/G \longrightarrow M & ; & [(q, q')] \longrightarrow p(q), \\
\tilde{\beta} : (Q \times Q)/G \longrightarrow M & ; & [(q, q')] \longrightarrow p(q'), \\
\tilde{\epsilon} : M \longrightarrow (Q \times Q)/G & ; & x \longrightarrow [(q, q)], \quad \text{if } p(q) = x, \\
\tilde{m} : ((Q \times Q)/G)_2 \longrightarrow (Q \times Q)/G & ; & ([(q, q')], [(gq', q'')]) \longrightarrow [(gq, q'')], \\
\tilde{i} : (Q \times Q)/G \longrightarrow (Q \times Q)/G & ; & [(q, q')] \longrightarrow [(q', q)].
\end{array}
$$

This Lie groupoid is called *the Atiyah (gauge) groupoid associated with the principal* G*-bundle* $p : Q \to M$.

$$(Q \times Q)/G \rightsquigarrow (TQ)/G$$

A. Weinstein: Lagrangian Mechanics and groupoids, Fields Inst. Comm. **7** *(1996), 207-231.*

J.C. Marrero, D. Martín de Diego, E. Martínez: Discrete Lagrangian and Hamiltonian Mechanics on Lie groupoids, Nonlinearity 2006.

Discrete Unconstrained Lagrangian Systems

$L_d : G \to \mathbb{R}$ Discrete lagrangian, $\qquad g \in G$

- *Admissible sequences*

$$\mathcal{C}_g^N\{(g_1, \ldots, g_N) \in G^N \,|\, (g_k, g_{k+1}) \in G_2 \text{ and } g_1 \ldots g_N = g\}$$

- *Infinitesimal variations*

$$T_{(g_1,\ldots,g_N)}\mathcal{C}_g^N = \{(v_1, \ldots, v_{N-1}) \,|\, v_k \in (AG)_{x_k} \text{ for } x_k = \beta(g_k)\}$$

- *Discrete action sum*

$$\mathcal{S}L_d(g_1, \ldots, g_N) = \sum_{k=1}^{N} L_d(g_k)$$

$(g_1, \ldots, g_N)$ a solution of the *discrete unconstrained Euler-Lagrange equations* for L_d if

$$\sum_{k=1}^{N-1} d^o[L_d \circ l_{g_k} + L_d \circ r_{g_{k+1}} \circ i](\epsilon(x_k))_{|(AG)_{x_k}} = 0$$

$$\beta(g_k) = \alpha(g_{k+1}) = x_k$$

The *discrete unconstrained Euler-Lagrange operator* $D_{DEL}L_d : G_2 \to AG^*$

$$(D_{DEL}L_d)(g, h) = d^o[L_d \circ l_g + L_d \circ r_h \circ i](\epsilon(x))_{|(AG)_x} = 0,$$

for $(g, h) \in G_2$, with $\beta(g) = \alpha(h) = x \in M$.

The Lie algebroid $\widetilde{\tau_G} : \mathcal{T}^G G \to G$

$\widetilde{\tau}_G : \mathcal{T}^G G \equiv V\beta \oplus_G V\alpha \to G$ Lie algebroid

$$(\rho^{\mathcal{T}^G G})(X_g, Y_g) = X_g + Y_g$$

$$[\![(\overrightarrow{X}, \overleftarrow{Y}), (\overrightarrow{X'}, \overleftarrow{Y'})]\!]^{\mathcal{T}^G G} = (-\overrightarrow{[\![X, X']\!]}, \overleftarrow{[\![Y, Y']\!]}), \text{ for } X, Y, X', Y' \in \mathrm{Sec}(\tau).$$

Discrete Poincaré-Cartan sections

- *Poincaré-Cartan 1-section* $\Theta^{+}_{L_d}$

$$\Theta^{-}_{L_d}(g)(X_g, Y_g) = -X_g(L_d),$$
$$\Theta^{+}_{L_d}(g)(X_g, Y_g) = Y_g(L_d),$$

for each $g \in G$ and $(X_g, Y_g) \in \mathcal{T}^G_g G \equiv V_g\beta \oplus V_g\alpha$.

- *Poincaré-Cartan 2-section* Ω_{L_d}

$$\Omega_{L_d} = d\Theta^{-}_{L_d} = -d\Theta^{+}_{L_d}$$

Discrete unconstrained Lagrangian evolution operator

Let $\Upsilon : G \to G$ be a smooth map such that:

- graph$(\Upsilon) \subseteq G_2$, that is, $(g, \Upsilon(g)) \in G_2$, for all $g \in G$
- $(g, \Upsilon(g))$ is a solution of the discrete unconstrained Euler-Lagrange equations, for all $g \in G$, that is, $(D_{DEL} L_d)(g, \Upsilon(g)) = 0$, for all $g \in G$.

$$\overleftarrow{X}(g)(L_d) - \overrightarrow{X}(\Upsilon(g))(L_d) = 0,$$

for every section $X \in Sec(\tau)$ and every $g \in G$.

Discrete Flow. Regularity

$\Upsilon : G \to G$ second order operator
$\mathcal{T}\Upsilon : \mathcal{T}^G G \to \mathcal{T}^G G$ Lie algebroid morphism over $\Upsilon : G \to G$

$$\begin{aligned} \mathcal{T}_g\Upsilon(X_g, Y_g) &= ((T_g(r_{g\Upsilon(g)} \circ i))(Y_g), (T_g\Upsilon)(X_g) \\ &+ (T_g\Upsilon)(Y_g) - T_g(r_{g\Upsilon(g)} \circ i)(Y_g)), \end{aligned}$$

for all $(X_g, Y_g) \in \mathcal{T}^G_g G \equiv V_g\beta \oplus V_g\alpha$.

The map Υ is a discrete unconstrained Lagrangian evolution operator for L_d if and only if $(\mathcal{T}\Upsilon, \Upsilon)^*\Theta^-_{L_d} = \Theta^+_{L_d}$.

Proposition ($\Upsilon : G \to G$ is a discrete flow)
If $\Upsilon : G \to G$ is a discrete unconstrained Lagrangian evolution operator then

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A discrete Lagrangian $L_d : G \to \mathbb{R}$ is *regular* if the Poincaré-Cartan 2-section Ω_{L_d} is symplectic.

$L_d : G \to \mathbb{R}$ be a regular discrete Lagrangian function and $(g_0, h_0) \in G \times G$ be a solution of the discrete E-L equations for L_d.

$\Rightarrow$ there exists a (local) discrete unconstrained Lagrangian evolution operator $\Upsilon_{L_d} : U_0 \to V_0$ such that

- $\Upsilon_{L_d}(g_0) = h_0$,
- Υ_{L_d} is a diffeomorphism and
- Υ_{L_d} is unique, that is, if U_0' is an open subset of G, with $g_0 \in U_0'$, and $\Upsilon'_{L_d} : U_0' \to G$ is a (local) discrete Lagrangian evolution operator then

$$\Upsilon_{L_d|U_0 \cap U_0'} = \Upsilon'_{L_d|U_0 \cap U_0'}.$$

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Discrete unconstrained Legendre transformations

$L_d : G \to \mathbb{R}$ discrete Lagrangian
The *discrete unconstrained Legendre transformations*
$\mathbb{F}^- L_d : G \to AG^*$ and $\mathbb{F}^+ L_d : G \to AG^*$

$$(\mathbb{F}^- L_d)(h)(v_{\epsilon(\alpha(h))}) = -v_{\epsilon(\alpha(h))}(L_d \circ r_h \circ i), \text{ for } v_{\epsilon(\alpha(h))} \in (AG)_{\alpha(h}$$
$$(\mathbb{F}^+ L_d)(g)(v_{\epsilon(\beta(g))}) = v_{\epsilon(\beta(g))}(L_d \circ l_g), \text{ for } v_{\epsilon(\beta(g))} \in (AG)_{\beta(g)}.$$

$$\begin{aligned} L_d \text{ is regular} \quad &\Longleftrightarrow \quad \mathbb{F}^+ L_d \text{ is a local diffeomorphism} \\ &\Longleftrightarrow \quad \mathbb{F}^- L_d \text{ is a local diffeomorphism} \end{aligned}$$

J.C. Marrero, D. Martín de Diego, E. Martínez: *Discrete Lagrangian and Hamiltonian Mechanics on Lie groupoids*, Nonlinearity 2006.

- We introduce two Poincaré-Cartan 1-sections Θ_L^+ and Θ_L^-, and a unique Poincaré-Cartan 2-section, Ω_L.
- We study the discrete Lagrangian evolution operator $\xi : G \longrightarrow G$ and its preservation properties.
- Reduction theory is established in terms of morphisms of Lie groupoids.
- The associated Hamiltonian formalism is developed using the discrete Legendre transformations $\mathbb{F}^+L : G \to A^*G$ and $\mathbb{F}^-L : G \to A^*G$.
- A complete characterization of the regularity of a Lagrangian on a Lie groupoid is given in terms of the symplecticity of Ω_L.
- We prove a Noether's theorem for discrete Mechanics on Lie groupoids.

As a concrete example of a system on a transformation Lie groupoid we consider a discretization of the heavy top. In the continuous theory (as Juan Carlos explained yesterday), the configuration manifold is the transformation Lie algebroid $\tau : S^2 \times \mathfrak{so}(3) \to S^2$ with Lagrangian

$$L_c(\Gamma, \Omega) = \frac{1}{2}\Omega \cdot I\Omega - mgl\Gamma \cdot \mathbf{e},$$

where $\Omega \in \mathbb{R}^3 \simeq \mathfrak{so}(3)$ is the angular velocity, Γ is the direction opposite to the gravity and $\mathbf{e}$ is a unit vector in the direction from the fixed point to the center of mass, all them expressed in a frame fixed to the body. The constants m, g and l are respectively the mass of the body, the strength of the gravitational acceleration and the distance from the fixed point to the center of mass. The matrix I is the inertia tensor of the body.

In order to discretize this Lagrangian it is better to express it in terms of the matrices $\hat{\Omega} \in \mathfrak{so}(3)$ such that $\hat{\Omega} v = \Omega \times v$. Then

$$L_c(\Gamma, \Omega) = \frac{1}{2} \operatorname{Tr}(\hat{\Omega} \mathbb{I} \hat{\Omega}^T) - mgl\Gamma \cdot \mathbf{e}.$$

where $\mathbb{I} = \frac{1}{2} \operatorname{Tr}(I) I_3 - I$.

We can define a discrete Lagrangian $L_d : G = S^2 \times SO(3) \to \mathbb{R}$ for the heavy top by

$$L_d(\Gamma_k, W_k) = -\frac{1}{h} \operatorname{Tr}(\mathbb{I} W_k) - hmgl\Gamma_k \cdot \mathbf{e}.$$

which is obtained by the rule $\hat{\Omega} = R^T \dot{R} \approx \frac{1}{h} R_k^T (R_{k+1} - R_k) = \frac{1}{h}(W_k - I_3)$, where $W_k = R_k^T R_{k+1}$.

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The value of the action on an admissible variation is

$$\lambda(t) = L_d(\Gamma_k, W_k e^{tK}) + L(e^{-tK}\Gamma_{k+1}, e^{-tK}W_{k+1})$$
$$= -\frac{1}{h}\left[\mathrm{Tr}(IW_k e^{tK}) + mglh^2\Gamma_k \cdot \mathbf{e} + \mathrm{Tr}(Ie^{-tK}W_{k+1}) + mglh^2(e^{-tK}\Gamma_{k+1}) \cdot \mathbf{e}\right],$$

where $\Gamma_{k+1} = W^T\Gamma_k$ (since the above pairs must be composable) and $K \in \mathfrak{so}(3)$ is arbitrary.

Taking the derivative at $t = 0$ and after some straightforward manipulations we get the DEL equations

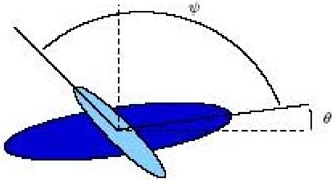
$$M_{k+1} - W_k^T M_k W_k + mglh^2(\widehat{\Gamma_{k+1} \times \mathbf{e}}) = 0$$

where $M = WI - IW^T$. Finally, in terms of the axial vector Π in $\mathbb{R}^3$ defined by $\hat{\Pi} = M$, we can write the equations in the form

$$\Pi_{k+1} = W_k^T\Pi_k + mglh^2\Gamma_{k+1} \times \mathbf{e}.$$

Example: Elroy's beanie

This system is probably the more simple example of a dynamical system with a non-Abelian Lie group symmetries. It consists in two planar rigid bodies attached at their centers of mass, moving freely in the plane.



Configuration space: The configuration space is $Q = SE(2) \times S^1$ with coordinates (x, y, θ, ψ), where the three first coordinates describe the position and orientation of the center of mass of the first body and the last one the relative orientation between both bodies.

Lagrangian function. We consider the Lagrangian

$$L(x, y, \theta, \psi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\psi}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2(\dot{\theta} + \dot{\psi})^2 - V(\psi)$$

where m denotes the mass of the system and I_1 and I_2 are the inertias of the first body and the second body, respectively; additionally, we also consider a potential function of the form $V(\psi)$.

Symmetry. The symmetry group we consider is $G = SE(2)$. The Lagrangian L is $SE(2)$-invariant.

Continuous system

$$l^R : TQ/SE(2) \longrightarrow \mathbb{R}$$

$$l^R(\psi, \dot\psi, \Omega) = \frac{1}{2} m(\Omega_1^2 + \Omega_2^2) + \frac{1}{2}(I_1 + I_2)\Omega_3^2 + \frac{1}{2}\frac{I_1 I_2}{I_1 + I_2}\dot\psi^2 - V(\psi)$$

Lagrange-Poincaré equations

$$\left\{ \begin{array}{rcl} \dot\Omega_1 & = & \Omega_2\Omega_3 - \frac{I_2}{I_1+I_2}\dot\psi\Omega_2 \\ \dot\Omega_2 & = & -\Omega_1\Omega_3 + \frac{I_2}{I_1+I_2}\dot\psi\Omega_1 \\ \dot\Omega_3 & = & 0 \\ \frac{I_1 I_2}{I_1+I_2}\ddot\psi & = & -\dfrac{\partial V}{\partial\psi} \end{array} \right\}$$

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Discrete system

We introduce the discrete Lagrangian l_d^R on $(Q \times Q)/SE(2)$ given by

$$l_d^R(\psi_k, \psi_{k+1}, \Omega_{(1)k}, \Omega_{(2)k}, \Omega_{(3)k}) = \frac{1}{2h^2} m \left[\Omega^2_{(1)k} + \Omega^2_{(2)k}\right] + \frac{I_1 + I_2}{h^2}\left[1 - \cos(\Omega_{(3)k})\right] + \frac{1}{2}\frac{I_1 I_2}{I_1 + I_2}\left(\frac{\Delta\psi_k}{h}\right)^2 - V\left(\frac{\psi_k + \psi_{k+1}}{2}\right)$$

Discrete Lagrange-Poincaré equations

$$\begin{cases} \Omega_{(1)k+1} = \Omega_{(1)k}\cos(\Omega_{(3)k} + \frac{I_2}{I_1+I_2}\Delta\Psi_k) - \Omega_{(2)k}\sin(\Omega_{(3)k} + \frac{I_2}{I_1+I_2}\Delta\Psi_k) \\ \Omega_{(2)k+1} = \Omega_{(1)k}\sin(\Omega_{(3)k} + \frac{I_2}{I_1+I_2}\Delta\Psi_k) + \Omega_{(2)k}\cos(\Omega_{(3)k} + \frac{I_2}{I_1+I_2}\Delta\Psi_k) \\ \Omega_{(3)k+1} = \Omega_{(3)k} \\ \frac{I_1 I_2}{I_1 + I_2}\frac{\psi_{k+2} - 2\psi_{k+1} + \psi_k}{h^2} = -\frac{1}{2}\left(\frac{\partial V}{\partial \psi}\left(\frac{\psi_{k+2} + \psi_{k+1}}{2}\right) + \frac{\partial V}{\partial \psi}\left(\frac{\psi_{k+1} + \psi_k}{2}\right)\right) \end{cases}$$

Discrete system

We introduce the discrete Lagrangian l_d^R on $(Q \times Q)/SE(2)$ given by

$$
l_d^R(\psi_k, \psi_{k+1}, \Omega_{(1)k}, \Omega_{(2)k}, \Omega_{(3)k}) = \frac{1}{2h^2} m \left[\Omega_{(1)k}^2 + \Omega_{(2)k}^2\right] + \frac{I_1 + I_2}{h^2}\left[1 - \cos(\Omega_{(3)k})\right] + \frac{1}{2}\frac{I_1 I_2}{I_1 + I_2}\left(\frac{\Delta\psi_k}{h}\right)^2 - V\left(\frac{\psi_k + \psi_{k+1}}{2}\right)
$$

Discrete Lagrange-Poincaré equations

$$
\left\{
\begin{array}{l}
\Omega_{(1)k+1} = \Omega_{(1)k} \cos(\Omega_{(3)k} + \frac{I_2}{I_1+I_2}\Delta\Psi_k) - \Omega_{(2)k} \sin(\Omega_{(3)k} + \frac{I_2}{I_1+I_2}\Delta\Psi_k) \\
\Omega_{(2)k+1} = \Omega_{(1)k} \sin(\Omega_{(3)k} + \frac{I_2}{I_1+I_2}\Delta\Psi_k) + \Omega_{(2)k} \cos(\Omega_{(3)k} + \frac{I_2}{I_1+I_2}\Delta\Psi_k) \\
\Omega_{(3)k+1} = \Omega_{(3)k} \\
\frac{I_1 I_2}{I_1 + I_2}\frac{\psi_{k+2} - 2\psi_{k+1} + \psi_k}{h^2} = -\frac{1}{2}\left(\frac{\partial V}{\partial \psi}\left(\frac{\psi_{k+2} + \psi_{k+1}}{2}\right) + \frac{\partial V}{\partial \psi}\left(\frac{\psi_{k+1} + \psi_k}{2}\right)\right)
\end{array}
\right.
$$

Discrete generalized Holder's principle

$G \rightrightarrows Q$ a Lie groupoid; $\dim G = m + n$, $\dim Q = m$

$\alpha, \beta : G \to Q$, $\epsilon : Q \to G; i : G \to G$, $m : G_2 \to G$

$\tau : AG \to Q$ the Lie algebroid of G

Generalized discrete nonholonomic (or constrained) Lagrangian system

- $L_d : G \to \mathbb{R}$ a *regular discrete Lagrangian*
- *The constraint distribution* $\mathcal{D}_c$
  $\tau_{\mathcal{D}_c} : \mathcal{D}_c \to Q$ a vector subbundle of AG, $\quad \text{rank}\mathcal{D}_c = r$
- *The discrete constraint embedded submanifold* $\mathcal{M}_c$
  $i_{\mathcal{M}_c} : \mathcal{M}_c \to G$ is a embedded submanifold of G

Assumption

$$\dim \mathcal{M}_c = \dim \mathcal{D}_c = m + r, \quad r \leqslant n$$

$(L_d, \mathcal{M}_c, \mathcal{D}_c) \equiv$ *a discrete nonholonomic Lagrangian system on* G

$g \in G$ fixed

$$\mathcal{C}_g^N = \{(g_1, \ldots, g_N) \in G^N / (g_k, g_{k+1}) \in G_2, \text{ for } k = 1, .., N-1 \text{ and } g_1 \ldots g_N = g\}$$

$$T_{(g_1, g_2, .., g_N)} \mathcal{C}_g^N \equiv \{(v_1, v_2, \ldots, v_{N-1}) \,|\, v_k \in (AG)_{x_k} \text{ and } x_k = \beta(g_k), 1 \leqslant k \leqslant N-1\}$$

<u>Discrete action sum</u>

$$\mathcal{S}L_d : \mathcal{C}_g^N \longrightarrow \mathbb{R} \quad (g_1, \ldots, g_N) \longmapsto \sum_{k=1}^{N} L_d(g_k)$$

$$(\mathcal{V}_c)_{(g_1, \ldots, g_N)} = \{(v_1, \ldots, v_{N-1}) \in T_{(g_1, \ldots, g_N)} \mathcal{C}_g^N / \forall k \in \{1, \ldots, N-1\}, \, v_k \in \mathcal{D}_c\}$$

Discrete Hölder's principle

Discrete Hölder's principle

$g \in G, \quad (g_1, \ldots, g_N) \in \mathcal{C}_g^N$

$(g_1, \ldots, g_N)$ is a solution of the discrete nonholonomic Lagrangian system: $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ if

$$\Updownarrow$$

- $g_k \in \mathcal{M}_c, \quad \forall k \in \{1, \ldots, N\}$
- $\delta \mathcal{S} L_d|_{(\mathcal{V}_c)_{g_1 \ldots, g_N}} = 0$

$$(g_1, \ldots, g_N) \in \mathcal{C}_g^N$$

$$\Downarrow$$

- $g_k \in \mathcal{M}_c, \quad \forall k \in \{1, \ldots, N\}$
- $\displaystyle\sum_{k=1}^{N-1} (d^o(L_d \circ l_{g_k}) + d^o(L_d \circ r_{g_{k+1}} \circ i))(\epsilon(\beta(g_k))_{|(\mathcal{D}_c)(\beta(g_k))} = 0$

$\beta(g_k) = \alpha(g_{k+1}) = x_k$

Discrete Nonholonomic equations

$$N = 2, \quad (g, h) \in G_2, \quad \beta(g) = \alpha(h) = x$$

$$(g, h) \text{ is a solution}$$

$$\Updownarrow$$

$$(g, h) \in \mathcal{M}_c \times \mathcal{M}_c, \qquad d^o(L_d \circ l_g + L_d \circ r_h \circ i)(\epsilon(x))_{|(\mathcal{D}_c)_x} = 0$$

Discrete nonholonomic Euler-Lagrange equations for the system $(L_d, \mathcal{M}_c, \mathcal{D}_c)$

D. Iglesias, J.C. Marrero, D. Martin de Diego and E. Martinez: Discrete Nonholonomic Lagrangian Systems on Lie Groupoids. To appear in Journal of Nonlinear Science (2008).

Alternative versions of the discrete nonholonomic E-L equations

$$\{X^\alpha\} \text{ a local basis of } \Gamma(\mathcal{D}_c^0)$$

$$(g,h) \in G_2, \quad \beta(g) = \alpha(h) = x \in Q$$

$$(g,h) \text{ is a solution of the discrete nonholonomic problem}$$

$$\Updownarrow$$

$$(g,h) \in \mathcal{M}_c \times \mathcal{M}_c$$

$$d^o \left[L_d \circ l_g + L_d \circ r_h \circ i \right] (\epsilon(x))(v) = \lambda_\alpha X^\alpha(x)(v),$$

$$\lambda_\alpha \equiv \text{ the Lagrange multipliers}$$

Alternative versions of the discrete nonholonomic E-L equations

$\{X_a\}$ a local basis of sections in $\Gamma(\mathcal{D}_c)$

$(g, h) \in \mathcal{M}_c \times \mathcal{M}_c, \quad \beta(g) = \alpha(h) = x \in Q$

(g, h) is a solution of the discrete nonholonomic problem

$$0 = \overleftarrow{X}_a\big|_{g_k}(L) - \overrightarrow{X}_a\big|_{g_{k+1}}(L), , \forall a$$

The standard case: $G = Q \times Q$

$$G = Q \times Q$$

$$(q_0, q_1) \in \mathcal{M}_c$$

$$((q_0, q_1), (q_1, q_2)) \text{ is a solution}$$

$$\Updownarrow$$

$$(q_1, q_2) \in \mathcal{M}_c$$

$$D_2 L_d(q_0, q_1) + D_1 L_d(q_1, q_2) = \lambda_\alpha A^\alpha(q_1)$$

Cortés, Martínez (2001)
McLachlan, Perlmuther(2006)

Discrete nonholonomic equations on Lie groups

$$g_1 \in \mathcal{M}_c$$

$(g_1, g_2) \in G \times G$ is a solution of the discrete nonholonomic Euler-Lagrange equations for $(L_d, \mathcal{M}_c, \mathcal{D}_c)$

$$\Updownarrow$$

$$g_1^{-1} dL_d(g_1) - dL_d(g_2) g_2^{-1} = \sum_{j=1}^{n-r} \lambda^j \mu_j,$$

$$g_1 \in \mathcal{M}_c$$

λ^j the Lagrange multipliers
$\{\mu_j\}$ a basis of $\mathcal{D}_c^0$

<u>Notation:</u> $g, h \in G$, $\alpha_h \in T_h^* G$

$$g\alpha_h = (T^*_{gh} l_{g^{-1}})(\alpha_h) \in T^*_{gh} G, \quad \alpha_h g = (T^*_{hg} r_{g^{-1}})(\alpha_h) \in T^*_{hg} G.$$

Federov, Zenkov (2005)
McLachLan, Perlmutter (2006)

An example of a discrete nonoholonomic Lagrangian systems on an Atiyah Lie groupoid.

"A (homogeneous) sphere of radius $r > 0$, mass m and inertia about any axis I rolls without sliding on a horizontal table which rotates with constant angular velocity Ω about a vertical axis through one of its points"

- Configuration space

$$Q = \mathbb{R}^2 \times SO(3), \qquad (x, y; R) \in Q$$

- The Lagrangian function $L : TQ \to \mathbb{R}$

$$L(x, y; R, \dot{x}, \dot{y}; \dot{R}) = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{4} I \mathrm{tr}(\dot{R}R^T(\dot{R}R^T)^T)$$

- The constrained submanifold $\mathcal{M}$

$$\mathcal{M} = \{(x, y; R, \dot{x}, \dot{y}; \dot{R}) / \qquad \begin{array}{l} \dot{x} + \frac{r}{2}\mathrm{tr}(\dot{R}R^T E_L) = -\Omega y \\ \dot{y} - \frac{r}{2}\mathrm{tr}(\dot{R}R^T E_1) = \Omega x \} \end{array}$$

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

the standard basis of $\mathfrak{so}(3)$

The constrained system is $SO(3)$-invariant

$$\Downarrow$$

$(L', \mathcal{M}')$ a constrained Lagrangian system on the corresponding Atiyah algebroid

$$E' \cong TQ/SO(3) \longrightarrow \mathbb{R}^2$$

- The vector bundle $E' = TQ/SO(3) \cong T\mathbb{R}^2 \times \mathfrak{so}(3) \longrightarrow \mathbb{R}^2$
- The anchor map $\rho' : E' \cong T\mathbb{R}^2 \times \mathfrak{so}(3) \longrightarrow T\mathbb{R}^2$ the projection over the first factor
- The Lie bracket on $Sec(E')$

$$[\![s'_3, s'_4]\!]' = s'_5, \quad [\![s'_4, s'_5]\!]' = s'_3, \quad [\![s'_5, s'_3]\!]' = s'_4,$$

$$\{s'_i\}_{i=1,\dots,5} \text{ the canonical basis of } Sec(E')$$

- The reduced Lagrangian function:

$$L'(x, y, \dot{x}, \dot{y}; w) = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{4} I tr(w^2)$$

- The reduced constraint submanifold

$$\mathcal{M}' = \{(x, y, \dot{x}, \dot{y}; w) / \dot{x} + \frac{r}{2} tr(wE_2) = -\Omega y, \dot{y} - \frac{r}{2} tr(wE_1) = \Omega x\}$$

Objective: To discretize the nonholonomic Lagrangian system $(L', \mathcal{M}')$ on the Atiyah algebroid $E' \cong T\mathbb{R}^2 \times \mathfrak{so}(3) \to \mathbb{R}^2$
The discrete Atiyah groupoid $G' \cong \mathbb{R}^2 \times \mathbb{R}^2 \times SO(3) \rightrightarrows \mathbb{R}^2$
The discrete Lagrangian function Using an approximation of the (local) inverse of the exponential map

$$\exp : \mathfrak{so}(3) \to SO(3)$$

$$\Downarrow$$

$$L'_d(x_0, y_0, x_1, y_1; W_1) = \frac{1}{2} m[(\frac{x_1 - x_0}{h})^2 + (\frac{y_1 - y_0}{h})^2] + \frac{I}{(2h)^2} tr(W_1)$$

The *discrete constraint submanifold* $\mathcal{M}'_c$

$$\begin{aligned}
\frac{x_1 - x_0}{h} + \frac{r}{2h}\mathrm{tr}(W_1 E_2) &= -\Omega\frac{y_1 + y_0}{2}, \\
\frac{y_1 - y_0}{h} - \frac{r}{2h}\mathrm{tr}(W_1 E_1) &= \Omega\frac{x_1 + x_0}{2},
\end{aligned}$$

The discrete constraint distribution

$$\mathcal{D}'_c = < \{s'_5, rs'_1 + s'_4, rs'_2 - s'_3\} >$$

The discrete constrained Euler-Lagrange equations for $(L'_d, \mathcal{M}'_c, \mathcal{D}'_c)$

$$
\begin{aligned}
\frac{x_2 - 2x_1 + x_0}{h^2} + \frac{I\Omega}{I + mr^2}\frac{y_2 - y_0}{2h} &= 0 \\
\frac{y_2 - 2y_1 + y_0}{h^2} - \frac{I\Omega}{I + mr^2}\frac{x_2 - x_0}{2h} &= 0 \\
\mathrm{tr}\left((W_1 - W_2)E_3\right) &= 0 \\
\frac{x_2 - x_1}{h} + \frac{r}{2h}\mathrm{tr}(W_2E_2) + \Omega\frac{y_2 + y_1}{2} &= 0, \\
\frac{y_2 - y_1}{h} - \frac{r}{2h}\mathrm{tr}(W_2E_1) - \Omega\frac{x_2 + x_1}{2} &= 0
\end{aligned}
$$

$(x_0, y_0, x_1, y_1; W_1)$ are known

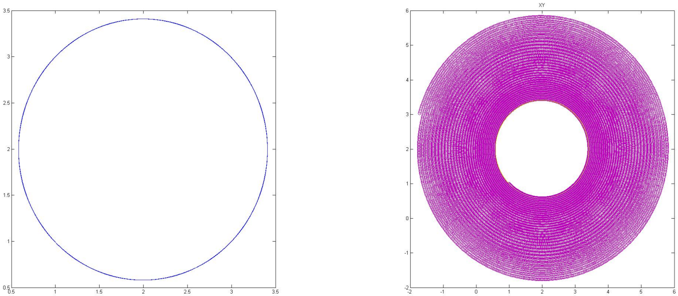

Figure: Orbits for the discrete nonholonomic equations of motion (left) and a standard numerical method (right) (initial conditions $x_0 = 0.99$, $y_0 = 1$, $x_1 = 1$, $y_1 = 0.99$ and $h = 0.01$ after 20000 steps).

Other numerical integrator for nonholonomic systems

S. Ferraro, D. Iglesias, D. Martín de Diego: Momentum and energy preserving integrators for nonholonomic dynamics, to appear in Nonlinearity.

Let $Gc : AG \times_M AG$ to$\mathbb{R}$ be a bundle metric on a Lie algebroid $(E, [\cdot, \cdot], \rho)$.

The class of systems that were considered is that of *mechanical systems with nonholonomic constraints* determined by:

- The Lagrangian function L:

$$L(a) = \frac{1}{2} Gc(a, a) - V(\tau(a)), \quad a \in AG,$$

  with V a function on M.

- The nonholonomic constraints determined by a subbundle $\mathcal{D}$ of AG,

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The class of systems that were considered is that of *mechanical systems with nonholonomic constraints* determined by:

- The Lagrangian function L:

$$L(a) = \frac{1}{2}\mathcal{G}c(a, a) - V(\tau(a)), \quad a \in AG,$$

  with V a function on M.

- The nonholonomic constraints determined by a subbundle $\mathcal{D}$ of AG, Consider the orthogonal decomposition $AG = \mathcal{D} \oplus \mathcal{D}^{\perp}$, and the associated orthogonal projectors

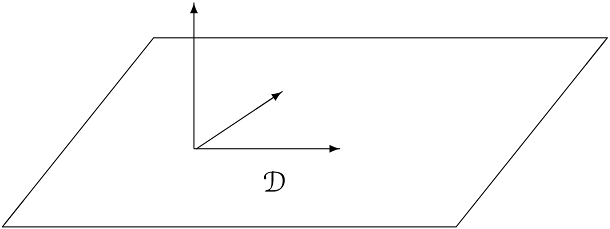
$$P: \; AG \longrightarrow \mathcal{D}$$
$$Q: \; AG \longrightarrow \mathcal{D}^{\perp}$$

Given local coordinates (x^i) in the base manifold M and a local basis of sections of AG (moving basis), $\{X_A\}$, adapted to the nonholonomic problem $(L, \mathcal{D})$, in the sense that

(i) $\{X_A\}$ is an orthonormal basis with respect to $\mathcal{G}c$ (that is $\mathcal{G}c(X_A, X_B) = \delta_{AB}$)

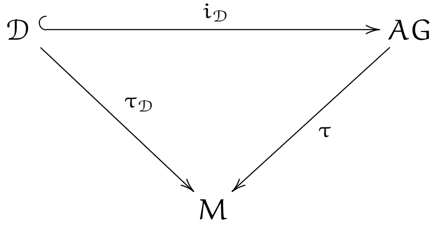
(ii) $\{X_A\} = \{X_a, X_\alpha\}$ where $\mathcal{D} = \text{span}\{X_a\}$, $\mathcal{D}^{\perp} = \text{span}\{X_\alpha\}$.

$\mathcal{D}^{\perp}$



$\mathcal{D}$

Denoting by $(x^i, y^A) = (x^i, y^a, y^\alpha)$ the induced coordinates on AG, the constraint equations determining $\mathcal{D}$ just read $\boxed{y^A = 0.}$ Therefore we choose (x^i, y^a) as a set of coordinates on $\mathcal{D}$.



In this coordinates we have the inclusion

$$\begin{array}{rccc} i_{\mathcal{D}}: & \mathcal{D} & \longrightarrow & AG \\ & (x^i, y^a) & \longmapsto & (x^i, y^a, 0) \end{array}$$

and the dual map

$$\begin{array}{rccc} i_D^*: & A^*G & \longrightarrow & D^* \\ & (x^i, y_a, y_\alpha) & \longmapsto & (x^i, y_a) \end{array}$$

where (x^i, y_A) are the induced coordinates on A^*G by the dual

Moreover from the orthogonal decomposition we have that

$$\begin{array}{rrcl} P: & AG & \longrightarrow & \mathcal{D} \\ & (x^i, y^a, y^\alpha) & \longmapsto & (x^i, y^a) \end{array}$$

and its dual map

$$\begin{array}{rrcl} P^*: & \mathcal{D}^* & \longrightarrow & A^*G \\ & (x^i, y_a) & \longmapsto & (x^i, y_a, 0) \end{array}$$

Nonholonomic Integrator

$L_d : G \longrightarrow \mathbb{R}$ discretization of L.

$$
\begin{aligned}
0 &= \overleftarrow{X}_a\Big|_{g_k}(L_d) - \overrightarrow{X}_a\Big|_{g_{k+1}}(L_d), \forall a \\
0 &= \overleftarrow{X}_\alpha\Big|_{g_k}(L_d) + \overrightarrow{X}_\alpha\Big|_{g_{k+1}}(L_d), \forall \alpha
\end{aligned}
$$