A presymplectic approach to Hamilton- Jacobi-Bellman equation

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CSIC

7th AIMS International Conference on Dyn. Systems, Diff. Equations and Applications
Special Session 44: Nonholonomic constraints in Mechanics and Optimal Control Theory
Arlington Texas USA May 18 - 21, 2008

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Classical Hamilton-Jacobi theory (geometric version)

The standard formulation of the Hamilton-Jacobi problem is to find a function $W(t, q^A)$ (called the principal function) such that

$$\frac{\partial W}{\partial t} + h(q^A, \frac{\partial W}{\partial q^A}) = 0. \qquad (1)$$

If we put $W(t, q^A) = S(q^A) - tE$, where E is a constant, then S satisfies

$$h(q^A, \frac{\partial S}{\partial q^A}) = E; \qquad (2)$$

S is called the characteristic function.

Equations (1) and (2) are indistinctly referred as the Hamilton-Jacobi equation.

R. Abraham, J.E. Marsden: *Foundations of Mechanics* **(2nd edition). Benjamin-Cumming, Reading, 1978.**

Let M be the configuration manifold, and T^*M its cotangent bundle equipped with the canonical symplectic form

$$\omega_M = dq^A \wedge dp_A$$

where (q^A) are coordinates in M and (q^A, p_A) are the induced ones in T^*M.

Let $H : T^*M \longrightarrow \mathbb{R}$ a hamiltonian function and X_H the corresponding hamiltonian vector field:

$$i_{X_H}\omega_M = dH$$

The integral curves of X_H, $(q^A(t), p_A(t))$, satisfy the Hamilton equations:

$$\frac{dq^A}{dt} = \frac{\partial H}{\partial p_A}\,, \quad \frac{dp_A}{dt} = -\frac{\partial H}{\partial q^A}$$

Let λ be a closed 1-form on M, say $d\lambda = 0$; (then, locally $\lambda = dS$)

Hamilton-Jacobi Theorem

The following conditions are equivalent:

(i) If $\sigma : I \to M$ satisfies the equation

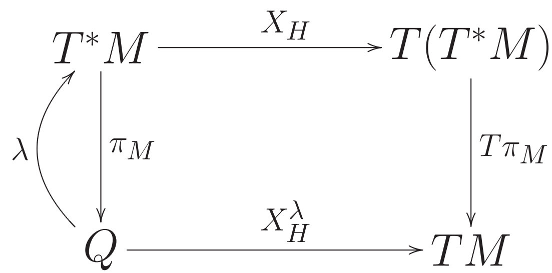
$$\frac{dq^A}{dt} = \frac{\partial H}{\partial p_A}$$

then $\lambda \circ \sigma$ is a solution of the Hamilton equations;

(ii) $d(H \circ \lambda) = 0$

Define a vector field on M:

$$X_H^\lambda = T\pi_M \circ X_H \circ \lambda$$



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(i) If $\sigma : I \to M$ satisfies the equation

$$\frac{dq^A}{dt} = \frac{\partial H}{\partial p_A}$$

then $\lambda \circ \sigma$ is a solution of the Hamilton equations;

(i)' If $\sigma : I \to M$ is an integral curve of X_H^λ, then $\lambda \circ \sigma$ is an integral curve of X_H;

(i)" X_H and X_H^λ are λ-related, i.e.

$$T\lambda(X_H^\lambda) = X_H \circ \lambda$$

Hamilton-Jacobi Theorem

Let λ be a closed 1-form on M. Then the following conditions are equivalent:

(i) X_H^λ **and** X_H **are** λ**-related;**

(ii) $d(H \circ \lambda) = 0$

If

$$\lambda = \lambda_A(q)\, dq^A$$

then the Hamilton-Jacobi equation becomes

$$H(q^A, \lambda_A(q^B)) = const.$$

and we recover the classical formulation when

$$\lambda_A = \frac{\partial S}{\partial q^A}$$

Optimal Control Theory

A control system of ordinary differential equations is usually given by

$$\dot{x}^i = \Gamma^i(x(t), u(t))$$

where

- x^i, $1 \leq i \leq n$ **are called the state variables**
- u^a, $1 \leq a \leq m$ **are called the control functions**

Consider the following optimal control problem: given initial and final states x_0 and x_f, the objective is to find a smooth curve $c(t) = (x(t), u(t))$ such that

- $x(t_0) = x_0, x(T_f) = x_f$**,**
- $c(t)$ **satisfies the control equation,**
- **and minimizes the functional**

$$\mathcal{I}(c) = \int_{t_0}^{t_f} L(x(t), u(t))\, dt$$

for some cost function $L = L(x, u)$**.**

In geometric terms we have a control fiber bundle

$$\pi : C \longrightarrow B$$

with fiber coordinates (x^i, u^a); Γ is a vector field along π:

$$\Gamma = \Gamma^i(x,u)\,\frac{\partial}{\partial x^i}$$

and L is a function $L : C \longrightarrow \mathbb{R}$.

Consider now the fiber product

$$\pi_0 : W_0 = C \times_B T^*B \longrightarrow B$$

with canonical projection

$$\pi_0(x^i, u^a, p_i) = (x^i)$$

where (x^i, u^a, p_i) are fibred coordinates in W_0. We also have two projections:

$$\Pi_1 : W_0 \longrightarrow C \ , \ \Pi_2 : W_0 \longrightarrow T^*B$$

expressed in local coordinates as

$$\begin{aligned} \Pi_1(x^i, u^a, p_i) &= (x^i, u^a) \\ \Pi_2(x^i, u^a, p_i) &= (x^i, p_i) \end{aligned}$$

We denote $\omega_0 = \Pi_2^* \omega_B$, where ω_B is the canonical symplectic form on T^*B. Therefore, we have that

$$\omega_0 = dx^i \wedge dp_i$$

is a presymplectic form with kernel

$$\ker \omega_0 = \langle \frac{\partial}{\partial u^a} \rangle$$

The Pontryaguin hamiltonian is the function on W_0 defined by

$$H_0(c, p) = \langle \Gamma(c), p \rangle - \Pi_1^* L$$

where $c \in C$ and $p \in T^*B$ are in the same fiber of W_0. Then, we have

$$H_0(x^i, u^a, p_i) = p_i \Gamma^i - L(x^i, u^a)$$

Consider the equation

$$i_X\,\omega_0 = dH_0 \tag{3}$$

Since ω_0 **is presymplectic, we can apply to Equation (3) the presymplectic algorithm which produces a sequence of constraint submanifolds**

$$\cdots W_k \hookrightarrow \cdots W_2 \hookrightarrow W_1 \hookrightarrow W_0$$

Notice that W_1 **is defined by the primary constraints**

$$\phi^a = p_j\,\frac{\partial \Gamma^j}{\partial u^a} - \frac{\partial L}{\partial u^a}$$

and a solution X **has the form**

$$X = \Gamma^i\,\frac{\partial}{\partial x^i} + U^a\,\frac{\partial}{\partial u^a} - \left(p_j\,\frac{\partial \Gamma^j}{\partial x^i} - \frac{\partial L}{\partial x^i}\right)\,\frac{\partial}{\partial p_i} \tag{4}$$

where $U^a(x,u,p)$ **are undetermined functions.**

In order to ensure that the vector field X be tangent to W_1 we need to verify the following tangency condition:

$$X(\phi^b) = \Gamma^i \frac{\partial \phi^b}{\partial x^i} + U^a \frac{\partial \phi^b}{\partial u^a} - \psi_i \frac{\partial \phi^b}{\partial p_i} = 0 \tag{5}$$

where

$$\psi_i = p_j \frac{\partial \Gamma^j}{\partial x^i} - \frac{\partial L}{\partial x^i}$$

From (5) we deduce that if the matrix

$$\left(\frac{\partial \phi^b}{\partial u^a} \right) \tag{6}$$

is regular then we can explicitly obtain the functions U^a, or, in other words, the presymplectic algorithm stabilizes at W_1, that is, $W_2 = W_1$. In this case we say that the optimal control problem given by (C, Γ, L) is regular. It should be noticed that under these condition, the solution X of Eq. (3) along W_1 is unique. Of course, a direct computations shows that the converse also holds, i.e if the algorithm stabilizes at W_1 then the problem is regular.

In the regular case, condition (6) implies that we can obtain u^a in terms of the rest of coordinates, say

$$u^a = \zeta^a(x^i, p_i) \tag{7}$$

We remark that $\omega_1 = (\omega_0)_{|W_1}$ is a symplectic form with canonical coordinates (x^i, p_i).

Otherwise, we should continue the algorithm and get the secondary and higher order constraints.

We will give an interpretation in the usual language of control theory. For short we only consider here the regular case. A simple computations shows that an integral curve $(x^i(t), u^a(t), p_i(t))$ of X satisfies the following system of differential equations

$$\begin{aligned} \dot{x}^i &= \Gamma^i(x,u) && (8) \\ \dot{u}^a &= U^a(x,u,p) && (9) \\ \dot{p}_i &= -\psi_i && (10) \end{aligned}$$

Therefore, Eq. (8) is just the control equation, and Eq. (10) can be equivalently written as

$$\dot{p}_i = -\frac{\partial H_0}{\partial x^i}$$

since

$$\psi_i = \frac{\partial H_0}{\partial x^i}$$

A singular example

Consider the optimal control problem determined by the following system of differential equations

$$\dot{x} = y, \dot{y} = z, \dot{z} = u$$

and the cost function

$$L(x, y, z, u) = \frac{1}{2}(x^2 - y^2 + z^2)$$

We have

$$\Gamma(x, y, z, u) = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + u\frac{\partial}{\partial z}$$

$$\omega_0 = dx \wedge dp_x + dy \wedge dp_y + dz \wedge dp_z$$

$$H_0(x, y, z, u, p_x, p_y, p_z) = p_x y + z p_y + u p_z - L$$

where (x, y, z, p_x, p_y, p_z) are fibred coordinates in T^*B.

We obtain that a solution X should be of the form

$$X = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + u\frac{\partial}{\partial z} + U\frac{\partial}{\partial u} + x\frac{\partial}{\partial p_x} - (p_x + y)\frac{\partial}{\partial p_y} + (p_y + z)\frac{\partial}{\partial p_z}$$

and the submanifold W_1 is locally defined by the constraint $p_z = 0$.

Continuing the process (X should be tangent to W_1) we have a new constraint $p_y - z = 0$, so that W_2 is defined by the constraints

$$p_z = 0, p_y - z = 0$$

Finally, we obtain the final constraint submanifold $W_f = W_3$ defined by

$$p_z = 0, p_y - z = 0, p_x + y_u = 0$$

In addition, the solution is uniquely defined on W_3:

$$X = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + u\frac{\partial}{\partial z} - (x + z)U\frac{\partial}{\partial u} + x\frac{\partial}{\partial p_x} - (p_x + y)\frac{\partial}{\partial p_y}$$

Along this section we assume that (C, Γ, L) is a regular optimal control problem.

Let $\gamma : B \longrightarrow W_0$ be a section of $\pi_0 : W_0 = C \times_B T^*B \longrightarrow B$. In local coordinates we have

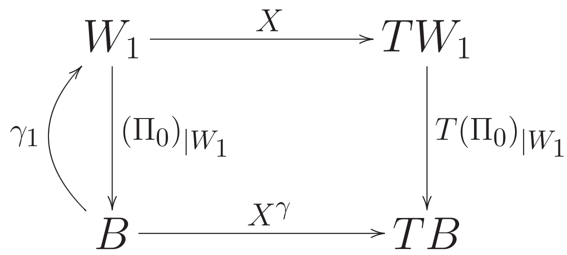
$$\gamma(x^i) = (x^i, \gamma^a(x), \gamma_i(x))$$

Notice that $\Pi_2 \circ \gamma$ is an ordinary 1-form on B.

Assume that $\gamma(B) \subset W_1$. Then we have

$$\phi^a(x^i, \gamma^a(x), \gamma_i(x)) = 0$$

Let $X : W_1 \longrightarrow TW_1$ be the solution of Eq. (3). Denoting by $\gamma_1 : B \longrightarrow W_1$ the restriction of γ we can construct a vector field on B such that the following diagram is commutative



Theorem

Let γ be a section of $\pi_0 : W_0 = C \times_B T^*B \longrightarrow B$ such that $\Pi_2 \circ \gamma$ is a closed 1-form on B and $\gamma(B) \subset W_1$. Then, the following conditions are equivalent:

(i) the vector fields X and X^γ are γ_1-related;

(ii) $d(H_0 \circ \gamma) = 0$

Proof: Let us recall the local expressions of X and X^γ:

$$\begin{aligned} X &= \Gamma^i \frac{\partial}{\partial x^i} + U^a \frac{\partial}{\partial u^a} - \left(p_j \frac{\partial \Gamma^j}{\partial x^i} - \frac{\partial L}{\partial x^i} \right) \frac{\partial}{\partial p_i} \\ X^\gamma &= \Gamma^i \frac{\partial}{\partial x^i} \end{aligned}$$

and then, after some straightforward computations, we have

$$X - T\gamma_1(X^\gamma) = -\left(\psi_i - \Gamma^i \frac{\partial \gamma_j}{\partial x^i} \right) \frac{\partial}{\partial p_j} \tag{11}$$

Also from a direct calculation we get

$$d(H_0 \circ \gamma) = \left(-\psi_i + \Gamma^j \frac{\partial \gamma_j}{\partial x^i} \right) dx^i \tag{12}$$

The result now follows from (11) and (12) taking into account that

$$\frac{\partial \gamma_i}{\partial x^j} = \frac{\partial \gamma_j}{\partial x^i}$$

since $\Pi_2 \circ \gamma$ is closed.

$\square$

The condition (ii) in the above theorem, can be equivalently written as

$$H_0(x^i, \zeta^a(x^j, \zeta^a(x^j, \gamma_j(x)), \gamma_i(x)) = constant \tag{13}$$

Since $\Pi_2 \circ \gamma$ **is closed, then it is locally exact, say** $\Pi_2 \circ \gamma = dS$**, where** S **is a function on** B**. Therefore, the equation**

$$H_0(x^i, \zeta^a(x^j, \frac{\partial S}{\partial x^j}), \frac{\partial S}{\partial x^i}) = constant \tag{14}$$

Notice that (14) is equivalent to

$$\Gamma^i \frac{\partial S}{\partial x^i} - L = constant \tag{15}$$

which is just the Hamilton-Jacobi-Bellman equation where $S = S(x)$ **is the Bellman value function.**

The Hamilton-Jacobi-Bellman equation: the singular case

Let (C, Γ, L) is a singular optimal control problem.

Therefore, we can obtain a final constraint submanifold W_f of W_0 where the equation

$$i_X \omega_0 = dH_0$$

has a solution.

Assume that, for each r, we have

- $B_r = \pi_0(W_f)$ is a submanifold of B,

- $\pi_0 : W_r \longrightarrow B_r$ is a submersion.

Given a section $\gamma : B \longrightarrow W_0$ of $\pi_0 : W_0 = C \times_B T^*B \longrightarrow B$ such that $\gamma(B) \subset W_f$, we can consider the restriction γ_f to W_f.

Now, if X is a solution of the equation

$$i_X \omega_0 = dH_0$$

we can construct the vector field X_f^γ on B as follows:

$$X_f^\gamma = T(\pi_0)_{|W_f} \circ X \circ \gamma_f$$

THEOREM

Let γ **be a section of** $\pi_0 : W_0 = C \times_B T^*B \longrightarrow B$ **such that** $\Pi_2 \circ \gamma$ **is a closed 1-form on** B **and** $\gamma(B) \subset W_f$**. If** X **is a solution of the equation** $i_X \omega_0 = dH_0$ **along** W_f**, then, the following conditions are equivalent:**

(i) $X - T\gamma_f(X_f^\gamma) \in \ker \omega_0$**;**

(ii) $d(H_0 \circ \gamma_f) = 0$

where γ_f **denotes the restriction of** γ **to** W_f**.**