A Generalization of the Poincaré-Cartan Integral Invariant for a Nonlinear Nonholonomic Dynamical System

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Outline

1. Constraints
2. Generalized Variations
3. The two viewpoints
4. Equations in independent Poincaré parameters
5. Poincaré-Hamilton equations of motion for the nonlinear nonholonomic dynamical system
6. Poincaré-Cartan Integral Invariant

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Holonomic and Nonholonomic Constraints

Nonholonomic Constraints

$$F(u_1, ..., u_{3N}; \dot{u}_1, ..., \dot{u}_{3N}; t) \begin{matrix} < \\ = \\ > \end{matrix} 0$$

Nonintegrable equations (or inequalities), imposes conditions on the velocity components (Example: Neimark and Fufaev, 1972, pp. 223–225)

Holonomic and Nonholonomic Constraints

Holonomic Constraints

If F is integrable

$$\frac{dG(u_1, ..., u_{3N}; t)}{dt} \begin{array}{c} < \\ = \\ > \end{array} 0$$

that is

$$H(u_1, u_2. \ldots, u_{3N}; t) \begin{array}{c} < \\ = \\ > \end{array} 0$$

Linear and Nonlinear Constraints

Linear Constraints if

$$F(u_1, ..., u_{3N}; \dot{u}_1, ..., \dot{u}_{3N}; t)$$

is Linear (Nonlinear) with respect to $\dot{u}$'s

Linear and Nonlinear Constraints

Nonlinear Constraints if

$$F(u_1, \ldots, u_{3N}; \dot{u}_1, \ldots, \dot{u}_{3N}; t)$$

is Linear (Nonlinear) with respect to $\dot{u}$'s

Define the actual displacement in time '*dt*' by the relation

$$dx_p = \dot{x}_p dt \qquad p = 1, 2, \cdots, n \tag{3.1}$$

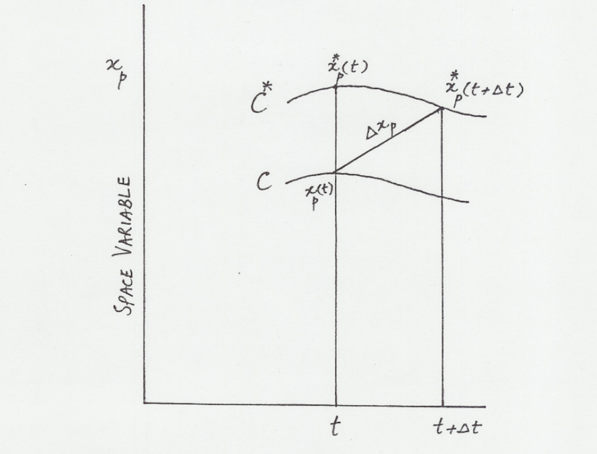
Any one of ∞^n configurations infinitely close to the actual trajectory and compatible with the constraints along a neighboring curve is defined by

$$x_p^*(t) = x_p(t) + \delta x_p \tag{3.2}$$

The quantities δx_p, arbitrary functions of the time t of class C^1 , are the variations of the variables $x_p(t)$ that are obtained by keeping the time t fixed. The difference

$$x_p^*(t) - x_p(t) = \delta x_p \tag{3.3}$$

is called the synchronous (or simultaneous) virtual variation of the variable $x_p(t)$ and is designated as δ-variation [Naseer 1985,Vujanovic 1978, 2004, Xu & Liu 1997]



x_p
SPACE VARIABLE
$\overset{*}{x}_p(t)$
$\overset{*}{x}_p(t+\Delta t)$
C^*
Δx_p
C
$x_p(t)$
t
$t+\Delta t$

Turning to a more general variation process, we define the position of the system in the actual (real) motion by the variables $x_p(t)$ and determine in a varied motion an infinitely closed position, by the function $x_p^*(t+\Delta t)$, where Δt is an infinitesimal change in time and is a differential function of the time t.

$$x^*(t+\Delta t) = x_p^*(t) + \dot{x}_p^*\Delta t = x_p(t) + \delta x_p + \dot{x}_p\Delta t, \tag{3.4}$$

Introducing the symbol Δx_p, we write

$$x^*(t+\Delta t) - x_p(t) = \Delta x_p = \delta x_p + \dot{x}_p\Delta t, \tag{3.5}$$

which is the variation Δx_p of the variable x_p where the time varies and the position along the varied path is not simultaneous to the actual path. These variations are called asynchronous (or non-simultaneous) virtual variations of the variable x_p and are indicated by the symbol Δ-variation. Further, the Δ-variation of an arbitrary function $F(x_p, t)$ of the class C^2 in the domain of the variables x_p and time t can be readily obtained as

$$\Delta F = \delta F + \dot{F}\Delta t. \tag{3.6}$$

$$d(\delta x_p)/dt = \delta \dot{x}_p \ ; \ (d(\Delta x_p)/dt = \Delta \dot{x}_p + \ddot{x}_p \Delta t), \tag{3.7}$$

which leads to the fact that the velocity of the variation and the variation of the velocity are the same (not the same) according to the process of δ-variation (Δ-variation), respectively.
We assume that the relation $d(\Delta t) = \Delta(dt)$ always holds for the independent variable t whereas it can be shown that $\Delta\delta x_p = \delta\Delta x_p$ holds for the dependent variables x_p.
In order to extend the existing theory of integral invariants we transform the preceding analysis to the Poincaré formalism that is based on the theory of continuous groups of transformations [Poincaré 1901].

Definition. The change dG of an arbitrary function $G(x_p, t)$, during the actual displacement dx_p in the time dt of the system, is determined by the relations [3]

$$dG = [X_0 G + \eta_p X_p G] dt \qquad (p = 1, 2, ..., n) \tag{3.8}$$

where X_0 and X_p, characterizing the infinitesimal displacement, are the operators defined by

$$X_0 = \frac{\partial}{\partial t} + \xi_0^q(x_p)\frac{\partial}{\partial x_q}, \quad X_p = \xi_p^q(x_r)\frac{\partial}{\partial x_q} \tag{3.9}$$

which form a transitive group of operators if we require that the commutators

$$(X_0, X_p) = X_0 X_p - X_p X_0, \quad (X_p, X_q) = X_p X_q - X_q X_p,$$

satisfy the relations

$$(X_0, X_p) = C_{0p}^q X_q, \quad (X_p, X_q) = C_{pq}^r X_r \qquad (p, q, r = 1, 2, ..., n) \tag{3.10}$$

Here C_{0p}^q, C_{pq}^r, which depend on the x's and the time t, are the structure constants corresponding to the operators X_0,X_p.

Definition In a simultaneous virtual displacement $\delta x_1, \delta x_2, ..., \delta x_n$ of the system, the change δG in an arbitrary function $G(x_p, t)$ is determined by the formula [7]:

$$\delta G = \omega_p X_p G. \qquad (p = 1, 2, ..., n) \tag{3.11}$$

Here the quantities ω_p are the parameters corresponding to the synchronous variation δG of the function and are called the Poincaré synchronous virtual displacement parameters or simply the virtual displacement parameters.

For the asynchronous variation ΔG of the function $G(x_p, t)$ during an infinitesimal time Δt, we substitute from (3.8) and (3.11) into the formula (3.6) to obtain

$$\Delta G = (\Delta t)X_0 G + (\omega_p + \eta_p \Delta t)X_p G$$

If the virtual displacements are characterized by the asynchronous variations Δx_p, then we need to extend the above definition. For this purpose,we use the notation $x_0 = t, \dot{\eta}_0 = 1$ and we write the expression (3.9) for the infinitesimal displacement operators in the compact form:

$$X_\mu = \xi_\mu^\nu(x_1, ..., x_n)\frac{\partial}{\partial x_\nu}; \quad \xi_0^0 = 1, \quad \xi_p^0 = 0 \qquad (\mu, \nu = 0, 1, 2, ..., n) \tag{3.12}$$

Further, we set

$$\omega_0 = 0, \quad \Delta x_0 = \Delta t = \Omega_0, \quad \dot{x}_0 = \eta_0 = 1 \tag{3.13}$$

and analogous to the Poincaré parameters ω_p [7], introduce the new parameters corresponding to asynchronous variations by the relation

$$\Omega_\mu = \omega_\mu + \eta_\mu \Omega_0 \tag{3.14}$$

which was first given in [1, 3].

Definition. The variation ΔG of an arbitrary function $G(x_p, t)$, during a virtual displacement $\Delta x_1, \Delta x_2, ..., \Delta x_n$ asynchronous to the time $\Delta t = \Omega_0$, is described by the formula

$$\Delta G = \Omega_\mu X_\mu G \qquad (\mu = 0, 1, ..., n) \tag{3.15}$$

where $\Omega_0, \Omega_1, ..., \Omega_n$, assumed to be functions of class C^2, are the parameters of the virtual displacement corresponding to the asynchronous variation. For future reference we shall call them the asynchronous virtual displacement parameters.
Moreover, we also get the compact form

$$dG = [\eta_\mu X_\mu G]dt \tag{3.16}$$

Since the system is holonomic, by virtue of relations (3.8),(3.9), (3.10) and the rule $\delta d = d\delta$, the synchronous variations $\delta\eta_p$ of the parameters of real displacement are

$$\delta\eta_p = \frac{d\omega_p}{dt} + C^p_{0q}\omega_q + C^p_{qr}\eta_q\omega_r \tag{3.17}$$

and, by means of the formula (3.6), it follows that

$$\Delta\eta_p = \delta\eta_p + \dot{\eta}_p\Omega_0 \tag{3.18}$$

Thus , using relations (3.14) and (3.17), it can readily be shown that

$$\Delta\eta_p = \dot{\Omega}_p - \eta_p\dot{\Omega}_0 + C^p_{qr}\eta_q\Omega_r + C^p_{0q}\Omega_q \tag{3.19}$$

which represents the asynchronous variations $\Delta\eta_p$ of the parameters of real displacement in terms of the new parameters Ω's of possible displacements.

Lemma. Let J be a functional defined by the integral

$$J = \int_0^t f dt, \tag{3.20}$$

where f is an arbitrary function of η_p, x_p and possibly the time t. Then the asynchronous variation ΔJ of the functional J is given by

$$\Delta J = \int_0^t (\Delta f + f\dot{\Omega}_0) dt. \tag{3.21}$$

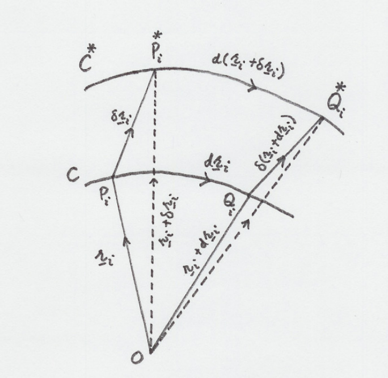
For the proof of the lemma, readers are referred to [3]. However, one can see that the non-commutativity of the Δ-operation and the integration is obvious due to the presence of the quantity $f\dot{\Omega}_0$.

Corollary. If the variation process is synchronous, then $\Delta t = \Omega_0 = 0$, which implies that Δ and δ are identical, hence $\int_0^t f \frac{d\Omega_0}{dt} dt$ vanishes. Which will imply

$$\delta \int_0^t f dt = \int_0^t \delta f \, dt$$

Transpositional Relation

$$\delta\eta_p = \frac{d\omega_p}{dt} + C^p_{0q}\omega_q + C^p_{qr}\eta_q\omega_r$$



Transpositional Relation

$$\delta\eta_p = \frac{d\omega_p}{dt} + C^p_{0q}\omega_q + C^p_{qr}\eta_q\omega_r$$

- From figure $\delta d\vec{r}_i = d\delta\vec{r}_i$
- $[\omega_p X_p X_0 r_i + \eta_p\omega_q X_q X_p r_i + \delta\eta_p X_p r_i]dt = d\omega_p X_p r_i + \omega_p(X_0 X_p r_i + \eta_q X_q X_p r_i)dt$
- which gives us $\delta\eta_p = \frac{d\omega_p}{dt} + C^p_{0q}\omega_q + C^p_{qr}\eta_q\omega_r$

The two viewpoints

Let us consider the dynamical system whose configuration is determined by the independent coordinates x_p and moves subject to the nonholonomic constraints expressed by the $(n-m)$ independent equations

$$f_\alpha(\eta_p, x_p, t) = 0 \qquad (p = 1, 2, ..., n; \alpha = m+1, ..., n), \tag{4.1}$$

where the functions f_α are not necessarily linear in the η_p's and that the virtual displacement parameters ω_p's satisfy the Chetaev's relations [7]

$$\frac{\partial f_\alpha}{\partial \eta_p}\omega_p = 0. \tag{4.2}$$

We further assume that the constraint equations (4.1) are expressible in the form

$$f_\alpha(\eta_p, x_p, t) = \eta_\alpha - \phi_\alpha(\eta_i, x_p, t) = 0, \tag{4.3}$$

$$(p = 1, 2, ..., n; i = 1, 2, ..., m < n; \alpha = m+1, ..., n)$$

From last two relations we get

$$\frac{\partial}{\partial \eta_p}(\eta_p - \phi_\alpha(x_p, \eta_p, t))\omega_p = \delta_{\alpha p}\omega_p - \frac{\partial \phi_\alpha}{\partial \eta_p}\omega_p = 0$$

which implies that the dependent parameters of virtual displacements satisfy

$$\omega_\alpha = \frac{\partial \phi_\alpha}{\partial \eta_p}\omega_p, \quad (p = 1, 2, \ldots, n; \alpha = m + 1, \ldots, n)$$

$$\omega_\alpha = \frac{\partial \phi_\alpha}{\partial \eta_j}\omega_j, \quad (j = 1, 2, \ldots, m < n; \alpha, \beta = m + 1, \ldots, n)$$

because $\frac{\partial \phi_\alpha}{\partial \eta_\beta} = 0$

- Denote $\eta_\alpha(\eta_i)$ and $\omega_\alpha(\omega_j)$ are dependent (independent) Poincaré parameters of real and virtual displacements respectively
- There are two viewpoints in using $\delta\eta_p = \frac{d\omega_p}{dt} + C^p_{0q}\omega_q + C^p_{qr}\eta_q\omega_r$

First Viewpoint about $\delta\eta_p = \frac{d\omega_p}{dt} + C^p_{0q}\omega_q + C^p_{qr}\eta_q\omega_r$

According to Hölder, Volterra and Hamel, above may be used for the δ-variation of all the parameters η_p of real displacement whether the system is holonomic or not. Thus the variations δf_α of f_α for the constraints are:

$$\delta f_\alpha = \left[X_p f_\alpha + (C^r_{0p} + C^r_{qp}\eta_q)\frac{\partial f_\alpha}{\partial \eta_r} - \frac{d}{dt}\left(\frac{\partial f_\alpha}{\partial \eta_p}\right)\right]\omega_p \qquad (4.4)$$

and

$$\delta f_\alpha = \delta\eta_\alpha - \delta\phi_\alpha = A^\alpha_i \omega_i \qquad (4.5)$$

Here the quantities A^α_i are determined by the relations

$$A^\alpha_i = \frac{d}{dt}\left(\frac{\partial \phi_\alpha}{\partial \eta_i}\right) - X_i\phi_\alpha - \frac{\partial \phi_\beta}{\partial \eta_i}X_\beta\phi_\alpha + (C^\alpha_{0i} + C^\alpha_{qi}\eta_q) + (C^\alpha_{0\beta} + C^\alpha_{q\beta}\eta_q)\frac{\partial \phi_\beta}{\partial \eta_i}$$
$$- \frac{\partial \phi_\alpha}{\partial \eta_j}\{(C^j_{0i} + C^j_{qi}\eta_q) + (C^j_{0\beta} + C^j_{q\beta}\eta_q)\frac{\partial \phi_\beta}{\partial \eta_i}\}$$
$$(q = 1, 2, ..., n; i, j = 1, 2, ..., m < n; \alpha, \beta = m + 1, ..., n)$$

Second Viewpoint about $\delta\eta_p = \frac{d\omega_p}{dt} + C^p_{0q}\omega_q + C^p_{qr}\eta_q\omega_r$

According to Amaldi, Levi-Civita and Suslov as discussed in [5], the above relation holds for nonholonomic systems. They may be used only for the variation $\delta\eta_i$ of the independent real parameters η_i together with the assumption that the δ-variation of the f_α vanishes. Precisely,

$$\delta\eta_i = \frac{d\omega_i}{dt} + C^i_{0q}\omega_q + C^i_{qr}\eta_q\omega_r \quad ; \quad \delta f_\alpha = 0 \tag{4.6}$$

for the constraints of the type (4.1).

The relations $\delta f_\alpha = 0$ enable us to obtain the synchronous virtual variations of the dependent η's. Thus, from (2.14) and (3.4), it follows that

$$\delta^* f_\alpha = \delta^* \eta_\alpha - \delta^* \phi_\alpha = (A_j^\alpha)^* \omega_j$$

Here $(A_j^\alpha)^*$ are given by

$$(A_j^\alpha)^* = \frac{d}{dt}\left(\frac{\partial \phi_\alpha}{\partial \eta_j}\right) - X_j^* \phi_\alpha + (K_{0j}^\alpha + K_{ij}^\alpha \eta_i + K_{\beta j}^\alpha \phi_\beta) - (K_{0j}^k + K_{ij}^k \eta_i + K_{\beta j}^k \phi_\beta)\frac{\partial \phi_\alpha}{\partial \eta_k}$$

where

$$\begin{aligned}
K_{0j}^k &= C_{0j}^k + C_{0\beta}^k \frac{\partial \phi_\beta}{\partial \eta_j} \quad ; \quad K_{0j}^\alpha = C_{0j}^\alpha + C_{0\beta}^\alpha \frac{\partial \phi_\beta}{\partial \eta_j} \\
K_{qj}^k &= C_{qj}^k + C_{q\beta}^k \frac{\partial \phi_\beta}{\partial \eta_j} \quad ; \quad K_{qj}^\alpha = C_{qj}^\alpha + C_{q\beta}^\alpha \frac{\partial \phi_\beta}{\partial \eta_j}
\end{aligned}$$

and

$$X_j^* = X_j + \frac{\partial \phi_\beta}{\partial \eta_j} X_\beta$$

Equations in independent Poincaré parameters

Consider a holonomic dynamical system described by $x_p(t)$, $p = 1, 2, \ldots, n$

- $$\left\{ \frac{d}{dt}\frac{\partial T}{\partial \eta_p} - C^q_{0p}\frac{\partial T}{\partial \eta_q} - C^r_{qp}\eta_q \frac{\partial T}{\partial \eta_r} - X_p T - Q_p \right\} \omega_p = 0$$
- If the system is conservative there exists $V(x_p)$ such that $Q_p = -X_p V$, then $\frac{\partial V}{\partial \eta_p} = 0$, above equation reduces to

$$\left\{ \frac{d}{dt}\frac{\partial T}{\partial \eta_p} - C^q_{0p}\frac{\partial T}{\partial \eta_q} - C^r_{qp}\eta_q \frac{\partial T}{\partial \eta_r} - X_p L \right\} \omega_p = 0$$

We begin by introducing the main ideas of Poincaré's formalism [7], [2]. Let us consider the dynamical system whose configuration is determined by the independent coordinates x_p and moves subject to the nonholonomic constraints expressed by the $(n-m)$ independent equations

$$f_\alpha(\eta_p, x_p, t) = 0 \qquad (p = 1, 2, ..., n; \alpha = m+1, ..., n), \tag{5.1}$$

where the functions f_α are not necessarily linear in the η_p's and that the virtual displacement parameters ω_p's satisfy the Chetaev's relations [7]

$$\frac{\partial f_\alpha}{\partial \eta_p}\omega_p = 0. \tag{5.2}$$

We further assume that the constraint equations (5.1) are expressible in the form

$$f_\alpha(\eta_p, x_p, t) = \eta_\alpha - \phi_\alpha(\eta_i, x_p, t) = 0, \tag{5.3}$$

$$(p = 1, 2, ..., n; i = 1, 2, ..., m < n; \alpha = m+1, ..., n)$$

From last two relations we get

$$\frac{\partial}{\partial \eta_p}(\eta_p - \phi_\alpha(x_p, \eta_p, t))\omega_p = \delta_{\alpha p}\omega_p - \frac{\partial \phi_\alpha}{\partial \eta_p}\omega_p = 0$$

which implies that the dependent parameters of virtual displacements satisfy

$$\omega_\alpha = \frac{\partial \phi_\alpha}{\partial \eta_p}\omega_p, \quad (p = 1, 2, \ldots, n; \alpha = m + 1, \ldots, n)$$

$$\omega_\alpha = \frac{\partial \phi_\alpha}{\partial \eta_j}\omega_j, \quad (j = 1, 2, \ldots, m < n; \alpha, \beta = m + 1, \ldots, n)$$

because $\frac{\partial \phi_\alpha}{\partial \eta_\beta} = 0$

- $\omega_\alpha(\omega_j)$ are dependent (independent)
- Previous equation does not yield the equation of motion
- Instead of using the multipliers we will use equation of nonholonomic constraints to express all quantities in terms of independent parameters ω_j's and η_j's.

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \eta_j}\right) - (K^k_{0j} + K^k_{\alpha j}\phi_\alpha + K^k_{ij}\eta_i)\frac{\partial L^*}{\partial \eta_k} - X^*_j L^* - (A^\alpha_j)^*\left(\frac{\partial L}{\partial \eta_\alpha}\right)^* = 0, \quad (5.4)$$

which are the required equations of motion independent of the Lagrange undetermined multipliers for the nonlinear nonholonomic dynamical system. These m equations determine the values of $\eta_j (j = 1, 2, ..., m)$ which, by virtue of the constraint equations (3.3), allow us to determine the values of the remaining $(n - m)$ η_α's as functions of the x_p's and t. The values of x_p are then calculated from the equations

$$\dot{x}_p = X_0 x_p + \eta_q X_q x_p \quad (5.5)$$

Poincaré-Hamilton equations of motion

Generalized momenta

$$y_p = \frac{\partial L}{\partial \eta_p}$$

Assume that transformation is invertible so that

$$\eta_p = \eta_p(x_q, y_q, t)$$

Above two relations yield, respectively, y's and η's as linear functions of η's and y's respectively. Since η's are not independent, it follows that y's are not independent as well.

Poincaré-Hamilton equations of motion

Introduce

$$y_j^* = \frac{\partial L^*}{\partial \eta_j}$$

which are assumed to be invertible and allow to express the η_j's as functions of y_j^*, x_p and time t in the form

$$\eta_j = \eta_j(x_p, y_j^*, t) \qquad (j = 1, 2, ..., m < n; p = 1, 2, ..., n) \tag{6.1}$$

Note that the constraint equations are expressed in terms of the independent y_j^*'s by means of the above equations.

Poincaré-Hamilton(PH) equations of motion

Definition. The $2m$ quantities $(x_1, ..., x_m; y_1^*, ..., y_m^*)$ are said to form a $2m$-dimensional reduced phase space provided that they are independent and satisfy the conditions

$$\delta t = 0; \quad \omega_j \neq 0; \quad \delta\eta_j \neq 0; \quad \delta y_j^* \neq 0 \tag{6.2}$$

throughout the motion of the dynamical system under the nonlinear nonholonomic constraints.

Poincaré-Hamilton equations of motion

Let us consider the motion of the system in this $2m$-dimensional reduced phase space and introduce the Hamiltonian function $H^*(x_p, y_j^*, t)$ corresponding to Lagrangian L^*

$$H^*(x_p, y_j^*, t) = \eta_j y_j^* - L^*(x_p, \eta_j, t) \tag{6.3}$$

Performing the δ-variation, using equations of motion and second viewpoint we get

$$\eta_j = \frac{\partial H^*}{\partial y_j^*}; \quad \dot{y}_j^* = -X_j^* H^* + (K_{0j}^k + K_{qj}^k \eta_q)\frac{\partial L^*}{\partial \eta_k} - (A_j^\alpha)^*(\frac{\partial L}{\partial \eta_\alpha})^* \tag{6.4}$$
$$(j, k = 1, 2, ..., m; \alpha = m+1, ..., n; q = 1, 2, ..., n)$$

These equations are called Poincaré-Hamilton equations of motion for the nonlinear nonholonomic dynamical system.

They together with the constraint equations determine the $(n+m)$ quantities $x_1, x_2, ..., x_n; y_1, y_2, ..., y_m$. In fact, we can find $\eta_j (j = 1, 2, ..., m)$ as functions of $x_p,\ y_j$ and t and then substituting the values of η_j into the constraints equations, determine the remaining $\eta_\alpha (\alpha = m + 1, ..., n)$. In this way all the η_j's are determined and then, by using $\dot{x}_p = X_0 x_p + \eta_q X_q x_p$, all the x_p's can be obtained as functions of the time t.

Poincaré-Cartan Integral Invariant

We need to compute the asynchronous variation of the action integral defined by

$$S = \int_{t_1}^{t_2} L dt \tag{7.1}$$

where Ldt expresses the small element of action with L, describing the dynamical behavior of the system, as the Lagrangian function of all the x's, η's and possibly the time t.

Poincaré-Cartan Integral Invariant

$$\Delta S = \int_{t_1}^{t_2} (\Delta L + L\dot{\Omega}_0) dt$$

which, by virtue of (A.4) from the APPENDIX, assumes the form

$$\begin{aligned} \Delta S &= \left(\frac{\partial L^*}{\partial \eta_j}\omega_j\right)\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \eta_p}\dot{\eta}_p\Omega_0 + \eta_\mu \Omega_0 X_\mu L - \Omega_0 \dot{L}\right) dt \\ &+ \int_{t_1}^{t_2} (L\Omega_0)^{\cdot} dt - \int_{t_1}^{t_2} \left[\frac{d}{dt}\left(\frac{\partial L^*}{\partial \eta_j}\right) - (K_{0j}^k + K_{qj}^k \eta_q)\frac{\partial L^*}{\partial \eta_k} - X_j^* L^* \right. \\ &- \left. (A_j^\alpha)^* \left(\frac{\partial L}{\partial \eta_\alpha}\right)^*\right] \omega_j dt \qquad (7.2) \end{aligned}$$

where the superscript " $*$ " shows that the quantities are expressed in terms of the independent Poincaré parameters of real displacement.

Poincaré-Cartan Integral Invariant

Theorem

The line integral

$$I = \oint_C (y_j^* \Omega_j - H^* \Omega_0), \tag{7.3}$$

along an arbitrary closed curve C remains invariant with arbitrary deformation of this curve along the tube of real trajectories of a conservative nonlinear nonholonomic dynamical system whose motion is governed by the equations of motion provided that the relations $\delta\eta_i = \frac{d\omega_i}{dt} + C^i_{0q}\omega_q + C^i_{qr}\eta_q\omega_r;\ \delta f_\alpha = 0$ *according to second viewpoint hold.*

Poincaré-Cartan Integral Invariant

Theorem

If the line integral (7.3) *is invariant under a deformation of an arbitrary curve C along the tube of real trajectories, then the motion of the conservative nonlinear nonholonomic dynamical system is determined by the Poincaré-Hamilton equations* (6.4) *together with* (5.5),*provided that the relations* (4.6) *hold.*

It is remarkable to note that the above two Theorems furnish the necessary and sufficient condition which allow us to connect the theory of integral invariants with the theory of Poincaré-Hamiltonian systems.

Poincaré-Cartan Integral Invariant

Theorem

The line integral

$$I_1 = \oint_C y_j^* \omega_j \tag{7.4}$$

along any closed curve C that consists of the simultaneous states of the system does not change with arbitrary deformation of this curve along the tube of real trajectories of the conservative nonlinear nonholonomic dynamical system which is described by the PH system of equations (6.4) *with* (5.5).

Poincaré-Cartan Integral Invariant

Theorem

If the line integral I_1, given by (7.3), remains invariant under an arbitrary deformation along the tube of real trajectories of any closed curve C consisting of the simultaneous states of a conservative nonlinear nonholonomic dynamical system, then the motion of the system is determined by the PH system of equations (6.4) with (5.5).

We remark that the results above show that the theory of integral invariants forms another basis for both the Hamiltonian dynamics of holonomic systems, linear nonholonomic systems and for nonlinear nonholonomic dynamical systems as well.

Special Cases:

(i) Suppose that all the x's are the Lagrangian coordinates and the η's are the generalized velocities $\dot{x}$'s. In this case, the relations (3.14) reduce to the result given in [Vujanovic] and [Whittaker] by Vujanovic and Whittaker, respectively; the operators X_0 and X_p's becomes $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x_p}$'s. Consequently all the C^p_{0q}'s and C^r_{qp}'s vanish. Last two theorems furnish the results analogous to those obtained in [Cartan], [Gantmacher], [Pars], and [Whittaker], while Theorems 7.1 and 7.2 subsume the results that are discussed in [Poincare, 1901, 1892] by Poincaré and in [1] by Pars. It is to be noted that our results are analogous to these but the content is quite different, since our system is nonlinear nonholonomic.

(ii) If the group variables are the quasi-variables (nonholonomic coordinates) π's then the η's becomes $\dot{\pi}$'s and the relations (2.10) express these as non-integrable linear combinations of the quasi-velocities and all the C^r_{qp}'s reduce to Hamel-Boltzmann's three indexed symbols γ^r_{pq}. In this case our theorems subsume the results analogous to those obtained by Djukic.

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Thank you

Performing the Δ-variation of (5.1) according to the Lemma 2.4, we have

$$\Delta S = \int_{t_1}^{t_2} (\Delta L + L\dot{\Omega}_0)dt$$
$$= \int_{t_1}^{t_2} (\frac{\partial L}{\partial \eta_p}\Delta\eta_p + \Omega_\mu X_\mu L + (L\Omega_0)\dot{} - \dot{L}\Omega_0)dt$$

where we have used Def. 2.3 and the identity $(L\Omega_0)\dot{} = L\dot{\Omega}_0 + \dot{L}\Omega_0$. Taking into account (2.16), (2.17) and (2.21), the variation ΔS of S becomes

$$\Delta S = \int_{t_1}^{t_2} (\frac{\partial L}{\partial \eta_p}\delta\eta_p + \omega_p X_p L)dt + \int_{t_1}^{t_2} (\frac{\partial L}{\partial \eta_p}\dot{\eta}_p\Omega_0 + \eta_\mu\Omega_0 X_\mu L - \Omega_0\dot{L})dt + \int_{t_1}^{t_2} (L\Omega_0)\dot{}dt \quad \text{(A-1)}$$

Let

$$I = \int_{t_1}^{t_2} (\frac{\partial L}{\partial \eta_p}\delta\eta_p + \omega_p X_p L)dt,$$

separating the sum over the index p from 1 to n into the sums over j from 1 to m and over α from $(m+1)$ to n and using the equation of motion (4.1) to get

$$I = \int_{t_1}^{t_2} \left\{ \left(\frac{\partial L^*}{\partial \eta_j} - \frac{\partial L}{\partial \eta_\alpha} \frac{\partial \phi_\alpha}{\partial \eta_j} \right) \delta\eta_j + \frac{\partial L}{\partial \eta_\alpha} \delta\eta_\alpha + \omega_p X_p L \right\} dt.$$

Here the '$*$' over the quantities show that they are expressed in terms of the independent parameters η_j of real displacement.
Following the second viewpoint and using relation (2.20), we have

$$I = \int_{t_1}^{t_2} \frac{\partial L^*}{\partial \eta_j} (\dot{\omega}_j + C^j_{0q}\omega_q + C^j_{qr}\eta_q\omega_r) dt - \int_{t_1}^{t_2} \frac{\partial L}{\partial \eta_\alpha} \frac{\partial \phi_\alpha}{\partial \eta_j} (\dot{\omega}_j + C^j_{0q}\omega_q + C^j_{qr}\eta_q\omega_r) dt + \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial \eta_\alpha} (\dot{\omega}_\alpha + C^\alpha_{0q}\omega_q + C^\alpha_{qr}\eta_q\omega_r) + \omega_p X_p L \right\} dt.$$

Integrating by parts, the first term of each integral on the right hand side of the last result, we get

$$
\begin{aligned}
I = & \left(\frac{\partial L^*}{\partial \eta_j}\omega_j\right)\Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L^*}{\partial \eta_j}\right)\omega_j dt + \int_{t_1}^{t_2} \frac{\partial L^*}{\partial \eta_j}(C^j_{0q}\omega_q + C^j_{qr}\eta_q\omega_r)dt \\
& - \left(\frac{\partial L}{\partial \eta_\alpha}\frac{\partial \phi_\alpha}{\partial \eta_j}\omega_j\right)\Bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \omega_j \frac{d}{dt}\left(\frac{\partial L}{\partial \eta_\alpha}\frac{\partial \phi_\alpha}{\partial \eta_j}\right)dt - \int_{t_1}^{t_2} \frac{\partial L}{\partial \eta_\alpha}\frac{\partial \phi_\alpha}{\partial \eta_j}(C^j_{0q}\omega_q + C^j_{qr}\eta_q\omega_r)dt \\
& + \left(\frac{\partial L}{\partial \eta_\alpha}\omega_\alpha\right)\Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \eta_\alpha}\right)\omega_\alpha dt + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \eta_\alpha}(C^\alpha_{0q}\omega_q + C^\alpha_{qr}\eta_q\omega_r) + \omega_p X_p L\right) dt
\end{aligned}
$$

which by use of (3.4), reduces to

$$
\begin{aligned}
I \quad = \quad & \left(\frac{\partial L^*}{\partial \eta_j}\omega_j\right)\Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left\{ \frac{d}{dt}\left(\frac{\partial L^*}{\partial \eta_j}\right)\omega_j - \frac{\partial L^*}{\partial \eta_j}(C^j_{0q}\omega_q + C^j_{qr}\eta_q\omega_r) \right. \\
- \quad & \omega_j \frac{\partial L}{\partial \eta_\alpha}\frac{d}{dt}\left(\frac{\partial \phi_\alpha}{\partial \eta_j}\right) - \omega_j \frac{d}{dt}\left(\frac{\partial L}{\partial \eta_\alpha}\right)\frac{\partial \phi_\alpha}{\partial \eta_j}\Big) + \frac{\partial L}{\partial \eta_\alpha}\frac{\partial \phi_\alpha}{\partial \eta_j}(C^j_{0q}\omega_q + C^j_{qr}\eta_q\omega_r) \\
+ \quad & \left. \frac{d}{dt}\left(\frac{\partial L}{\partial \eta_\alpha}\right)\frac{\partial \phi_\alpha}{\partial \eta_j}\omega_j - \frac{\partial L}{\partial \eta_\alpha}(C^\alpha_{0q}\omega_q + C^\alpha_{qr}\eta_q\omega_r) - \omega_p X_p L \right\} dt
\end{aligned}
$$

Again breaking the sum over the indices q and r from 1 to n into the sums over the indices j, k from 1 to m and over the indices

α, β from $(m+1)$ to n, the last expression takes the form

$$\begin{aligned} I &= \left(\frac{\partial L^*}{\partial \eta_j}\omega_j\right)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \Big\{ \frac{d}{dt}(\frac{\partial L^*}{\partial \eta_j})\omega_j - \frac{\partial L^*}{\partial \eta_j}(C^j_{0k}\omega_k + C^j_{0\alpha}\omega_\alpha + C^j_{qk}\eta_q\omega_k \\ &+ C^j_{q\alpha}\eta_q\omega_\alpha) - \omega_j \frac{\partial L}{\partial \eta_\alpha}\frac{d}{dt}(\frac{\partial \phi_\alpha}{\partial \eta_j}) + \frac{\partial L}{\partial \eta_\alpha}\frac{\partial \phi_\alpha}{\partial \eta_j}(C^j_{0k}\omega_k + C^j_{0\beta}\omega_\beta + C^j_{qk}\eta_q\omega_k \\ &+ C^j_{q\beta}\eta_q\omega_\beta) - \frac{\partial L}{\partial \eta_\alpha}(C^\alpha_{0j}\omega_j + C^\alpha_{0\beta}\omega_\beta + C^\alpha_{qj}\eta_q\omega_j + C^\alpha_{q\beta}\eta_q\omega_\beta) - \omega_p X_p L\Big\}\, dt \end{aligned}$$

Interchanging the indices j and k in the second and fourth terms of the integrand, using (3.4) and rearranging the terms, we find that

$$\begin{aligned} I = &\left(\frac{\partial L^*}{\partial \eta_j}\omega_j\right)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \Big[\frac{d}{dt}(\frac{\partial L^*}{\partial \eta_j})\omega_j - \frac{\partial L^*}{\partial \eta_k}\Big\{(C^k_{0j} + C^k_{0\alpha}\frac{\partial \phi_\alpha}{\partial \eta_j}) + \eta_q(C^k_{qj} + C^k_{q\alpha}\frac{\partial \phi_\alpha}{\partial \eta_j})\Big\}\omega_j \\ &- \omega_j\frac{\partial L}{\partial \eta_\alpha}\frac{d}{dt}(\frac{\partial \phi_\alpha}{\partial \eta_j}) + \frac{\partial L}{\partial \eta_\alpha}\frac{\partial \phi_\alpha}{\partial \eta_k}\Big\{(C^k_{0j} + C^k_{0\beta}\frac{\partial \phi_\beta}{\partial \eta_j}) + (C^k_{qj} + C^k_{q\beta}\frac{\partial \phi_\beta}{\partial \eta_j})\eta_q\Big\}\omega_j \\ &- \frac{\partial L}{\partial \eta_\alpha}\Big\{(C^\alpha_{0j} + C^\alpha_{0\beta}\frac{\partial \phi_\beta}{\partial \eta_j}) + (C^\alpha_{qj} + C^\alpha_{q\beta}\frac{\partial \phi_\beta}{\partial \eta_j})\eta_q\Big\}\omega_j - \omega_p X_p L\Big]\, dt \end{aligned}$$

Taking into account equations (3.11) and (3.12), the last result reduces to

$$\begin{aligned} I &= \frac{\partial L^*}{\partial \eta_j}\omega_j\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \Big[\frac{d}{dt}(\frac{\partial L^*}{\partial \eta_j})\omega_j - \frac{\partial L^*}{\partial \eta_k}(K^k_{0j} + K^k_{qj}\eta_q)\omega_j \\ &- \omega_j(\frac{\partial L}{\partial \eta_\alpha})^*\Big\{\frac{d}{dt}(\frac{\partial \phi_\alpha}{\partial \eta_j}) - \frac{\partial \phi_\alpha}{\partial \eta_k}(K^k_{0j} + K^k_{qj}\eta_q) + (K^\alpha_{0j} + K^\alpha_{qj}\eta_q)\Big\}\omega_j \\ &- \omega_p X_p L\Big]\, dt \end{aligned} \tag{A-2}$$

where the "∗" over the quantities $(\frac{\partial L}{\partial \eta_\alpha})^*$ indicates that they are expressed in terms of the independent parameters η_j.
Let us consider the term $\omega_p X_p L$ which, in view of (4.4), can be written as

$$\begin{aligned}
\omega_p X_p L &= \omega_p X_p L^* - \omega_p \frac{\partial L}{\partial \eta_\alpha} X_p \phi_\alpha \\
&= \omega_j X_j L^* + \omega_\alpha X_\alpha L^* - \omega_j \frac{\partial L}{\partial \eta_\alpha} X_j \phi_\alpha - \omega_\beta \frac{\partial L}{\partial \eta_\alpha} X_\beta \phi_\alpha \\
&= \omega_j X_j L^* + \frac{\partial \phi_\alpha}{\partial \eta_j} \omega_j X_\alpha L^* - \omega_j \frac{\partial L}{\partial \eta_\alpha} X_j \phi_\alpha - \frac{\partial \phi_\beta}{\partial \eta_j} \omega_j \frac{\partial L}{\partial \eta_\alpha} X_\beta \phi_\alpha
\end{aligned}$$

where to obtain this result we have separated the sum over the index $p = 1, 2, \ldots, n$ into the sums over $j = 1, 2, \ldots, m$ and $\alpha = m + 1, \ldots, n$ and also used the relation (4.3). Simplifying the last expression, we get

$$\omega_p X_p L = \omega_j (X_j + \frac{\partial \phi_\alpha}{\partial \eta_j} X_\alpha) L^* - \omega_j \frac{\partial L}{\partial \eta_\alpha} (X_j + \frac{\partial \phi_\beta}{\partial \eta_j} X_\beta) \phi_\alpha ,$$

which together with (3.13), becomes

$$\omega_p X_p L = \omega_j X_j^* L^* - \omega_j (\frac{\partial L}{\partial \eta_\alpha})^* X_j^* \phi_\alpha , \tag{A-3}$$

where we have expressed all the quantities in terms of the independent parameters η_j's of real displacement. This allows us to write (A.1) as

$$\begin{aligned}
I &= \left(\frac{\partial L^*}{\partial \eta_j} \omega_j \right) \Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left[\frac{d}{dt} (\frac{\partial L^*}{\partial \eta_j}) - \frac{\partial L^*}{\partial \eta_k} (K_{0j}^k + K_{qj}^k \eta q) - X_j^* L^* \right. \\
&- (\frac{\partial L}{\partial \eta_\alpha})^* \left\{ \frac{d}{dt} (\frac{\partial \phi_\alpha}{\partial \eta_j}) - \frac{\partial \phi_\alpha}{\partial \eta_k} (K_{0j}^k + K_{qj}^k \eta q) + (K_{0j}^\alpha + K_{qj}^\alpha \eta q) - X_j^* \phi_\alpha \right\} \Bigg] \omega_j dt
\end{aligned}$$

$$I = \left(\frac{\partial L^*}{\partial \eta_j} \omega_j \right) \Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left[\frac{d}{dt} (\frac{\partial L^*}{\partial \eta_j}) - (K_{0j}^k + K_{qj}^k \eta_q) \frac{\partial L^*}{\partial \eta_k} - X_j^* L^* - (A_j^\alpha)^* (\frac{\partial L}{\partial \eta_\alpha})^* \right] \omega_j dt \tag{A-4}$$

which expresses the asynchronous variation of the action integral (7.1).