Strict abnormal extremals in nonholonomic and kinematic control systems

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Outline

1. Optimal Control Problem: nonholonomic mechanical systems versus kinematic systems
2. Pontryagin's Maximum Principle
3. Hamiltonian problems: nonholonomic versus kinematic
4. Example

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Nonholonomic control mechanical systems

Let (Q, g) be a Riemannian manifold, $\dim Q = n$.
Let ∇ be the associated Levi-Civita connection.

$$\nabla_{\dot{\gamma}}\dot{\gamma} = F \circ \gamma + \sum_{r=1}^{n-m} \lambda^r Z_r \circ \gamma + \sum_{s=1}^{m} u^s Y_s \circ \gamma \ , \quad \dot{\gamma} \in D \ , \quad (1)$$

- $\gamma : I \longrightarrow Q$ is a differentiable curve;
- the *nonholonomic distribution* $D = \langle Y_1, \ldots, Y_m \rangle$, *input control vector fields*;
- $u^i : TQ \rightarrow U \subset \mathbb{R}^m$ are the *controls*;
- $D^\perp = \langle Z_1, \ldots, Z_{n-m} \rangle$;
- $F \in \mathfrak{X}(Q)$ is an *external force vector field*.

A *nonholonomic control mechanical system* is

$$\Sigma = (Q, g, F, D).$$

Equivalently to (1), $\dot{\gamma}$ is an integral curve of

$$Y = Z_g + F^V + \sum_{r=1}^{n-m} \lambda^r Z_r^V + \sum_{s=1}^{m} u^s Y_s^V$$

- $Z \in \mathfrak{X}(TQ)$ is the *geodesic spray* associated to g.
  In natural coordinates (x, v) for TQ,

$$Z = v^i \frac{\partial}{\partial x^i} - \Gamma^i_{jl}(x) v^j v^l \frac{\partial}{\partial v^i}, \quad \Gamma^i_{jl} \text{ Christoffel symbols for } \nabla .$$

- Y_k^V denotes the vertical lift of the vector field Y_k.

Kinematic control systems

The *kinematic system associated to (1)* is

$$\dot{\gamma}(t) = \sum_{s=1}^{m} w^s(t) Y_s(\gamma(t)) \ ,$$

- $\gamma\colon I \longrightarrow Q$ is a differentiable curve;
- $w^i \in \mathbb{R} \longrightarrow V \subset \mathbb{R}^m$ are the *controls*.

In other words, γ is an integral curve of the vector field

$$X = \sum_{s=1}^{m} w^s Y_s.$$

Theorem (Bullo, Lewis, 2005)

Every fully actuated nonholonomic control system Σ is equivalent to the associated kinematic system.

Optimal control problems

$\mathcal{F}\colon TQ \times U \to \mathbb{R}$ the cost function for the mechanical system.
$\mathcal{G}\colon Q \times V \to \mathbb{R}$ the cost function for the kinematic system.

Problem

Given $x_0, x_f \in Q$, *find* $\begin{matrix}(\gamma, u)\colon I \to Q \times U \\ (\gamma, w)\colon I \to Q \times V\end{matrix}$ *such that*

1. *end-point conditions on* Q: $\gamma(t_0) = x_0,\ \gamma(t_f) = x_f$;
2. $\left.\begin{matrix}\dot\gamma \\ \gamma\end{matrix}\right\}$ *is an integral curve of* $\left\{\begin{matrix} Y,\ \ddot\gamma(t) = Y(\dot\gamma(t), u(t)) \\ X,\ \dot\gamma(t) = X(\gamma(t), w(t))\end{matrix}\right.$
3. $\left.\begin{matrix}(\dot\gamma, u) \\ \hline (\gamma, w)\end{matrix}\right\}$ *gives the minimum of* $\left\{\begin{matrix}\int_I \mathcal{F}(\dot\gamma(t), u(t))dt \\ \hline \int_I \mathcal{G}(\gamma(t), w(t))dt\end{matrix}\right.$

among all the curves satisfying 1 *and* 2, *respectively.*

Extended systems: $\widehat{Q} = \mathbb{R} \times Q$

Extended kinematic control system:

$$\dot{\widehat{\gamma}} = \mathcal{G} \frac{\partial}{\partial x^0}\bigg|_{\widehat{\gamma}} + \sum_{s=1}^{m} w^s Y_s \circ \widehat{\gamma}.$$

Extended nonholonomic mechanical control system:

$$\widehat{\nabla}_{\dot{\widehat{\gamma}}} \dot{\widehat{\gamma}} = \mathcal{F} \circ (\dot{\widehat{\gamma}}, u) \frac{\partial}{\partial x^0}\bigg|_{\dot{\widehat{\gamma}}} + F \circ \widehat{\gamma} + \sum_{r=1}^{n-m} \lambda^r Z_r \circ \widehat{\gamma} + \sum_{s=1}^{m} u^s Y_s \circ \widehat{\gamma}$$

- $\widehat{\gamma} \colon I \to \widehat{Q}$ is a differentiable curve;
- $\widehat{\nabla}$ is the extended Levi-Civita connection, all the new Christoffel symbols are equal to zero;
- $\pi_1 \circ \dot{\widehat{\gamma}} = \dot{\gamma} \in D$ with $\pi_1 \colon T\widehat{Q} = T\mathbb{R} \times TQ \to TQ$.

Any connection?

$$\text{Nonholonomic} \quad \left.\begin{aligned} \dot{x}^0 &= v^0 \\ \dot{v}^0 &= \mathcal{F} \end{aligned}\right\} \quad \text{versus} \quad \text{Kinematic} \quad \dot{x}^0 = \mathcal{G}\ .$$

In some sense,

$$\mathcal{G} = v^0 = \int \mathcal{F}.$$

Proposition

Let $\mathcal{G}\colon I \times Q \longrightarrow \mathbb{R}$. If $(\dot{\gamma}, u)$ is an optimal curve of a nonholonomic mechanical control system with cost function $\mathcal{F} = \partial\mathcal{G}/\partial t + v^i \partial\mathcal{G}/\partial x^i = \widehat{\mathrm{d}\mathcal{G}}\colon I \times TQ \longrightarrow \mathbb{R}$, then there exist $w\colon I \longrightarrow V$ such that (γ, w) is an optimal curve of the kinematic system with cost function $\mathcal{G}$.

What about equivalence?

Let $(\widetilde{\gamma}, \widetilde{w})$ be a trajectory of the kinematic control system s.t.

$$\int \mathcal{G}(t, \dot{\gamma}(t))dt = \int dt \int \mathcal{F}(t, \dot{\gamma}(t))dt <$$

$$\int dt \int \mathcal{F}(t, \dot{\tilde{\gamma}}(t))dt = \int \mathcal{G}(t, \dot{\tilde{\gamma}}(t))dt$$

Proposition

The time optimal control problem for a nonholonomic mechanical control system is equivalent to the time optimal control problem for the associated kinematic system.

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Pontryagin's Maximum Principle, PMP

$\mathcal{F} \in \mathcal{C}^\infty(Q \times U)$. For $u \in U$, Hamiltonian $H^u \colon T^*\widehat{Q} \to \mathbb{R}$:

$$H^u(\widehat{x}, \widehat{p}) = \langle \widehat{p}, \widehat{X}(\widehat{x}, u)\rangle = p_0 \mathcal{F}(x,u) + \sum_{i=1}^{m} p_i X^i(x,u), \quad X \in \mathfrak{X}(Q).$$

Theorem

Let $(\widehat{\gamma}, u) \colon I \to \widehat{Q} \times U$ be a solution of OCP with end-point conditions x_a, x_f. Then there exists $(\widehat{\sigma}, u) \colon I \to T^*\widehat{Q} \times U$, with fiber momenta coordinates $\widehat{\lambda}(t) \in T^*_{\widehat{\gamma}^*(t)} Q$ such that:

1. *$(\widehat{\sigma}, u)$ is an integral curve of the Hamiltonian vector field X_{H^u}, $i_{X_{H^u}}\Omega = \mathrm{d}H^u$ and Ω the canonical 2−form on $T^*\widehat{Q}$;*
2. *$H(\widehat{\sigma}(t), u(t)) = \max_{\widetilde{u} \in U} H(\widehat{\sigma}(t), \widetilde{u})$ a. e.;*
3. *$\max_{\widetilde{u} \in U} H(\widehat{\sigma}(t), \widetilde{u}) = \text{constant}$ for $t \in I$;*
4. *$(\lambda_0, \lambda(t)) \neq 0$ for each $t \in I$ and λ_0* **is constant**.

Extremals

Definition

A curve $(\widehat{\gamma}, u)\colon I \to \widehat{Q} \times U$ *for* $\widehat{OCP}$ *is*

1. *an* **extremal** *if there exist* $\widehat{\sigma}\colon I \to T^*\widehat{Q}$ *such that* $\widehat{\gamma} = \pi_{\widehat{Q}} \circ \widehat{\sigma}$ *and* $(\widehat{\sigma}, u)$ *satisfies the necessary conditions of PMP;*
2. *a* **normal extremal** *if it is an extremal with* $\lambda_0 = -1$*;*
3. *an* **abnormal extremal** *if it is an extremal with* $\lambda_0 = 0$*;*
4. *a* **strictly abnormal extremal** *if it is not a normal extremal, but it is abnormal.*

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Nonholonomic mechanical hamiltonian

ASSUMPTION: Null connection, no external force.

Hamiltonian function $H_m \colon T^*T\widehat{Q} \times U \to \mathbb{R}$,

$$H_m = p_0 v^0 + q_0 \mathcal{F} + p_i v^i + \sum_{s=1}^{m} q_i u^s Y_s^i$$

and Hamilton's equations

$$\begin{array}{ll}
\dot{x}^0 = v^0 & \dot{p}_0 = 0 \\
\dot{x}^i = v^i & \dot{p}_i = -q_0 \frac{\partial \mathcal{F}}{\partial x^i} - q_j u^s \frac{\partial Y_s^j}{\partial x^i} \\
\dot{v}^0 = \mathcal{F} & \dot{q}_0 = -p_0 \\
\dot{v}^i = u^s Y_s^i & \dot{q}_i = -p_i
\end{array}$$

Extremals for the nonholonomic mechanical system

$$H_m = p_0 v^0 + q_0 \mathcal{F} + \dots$$

$$\dot{p}_0 = 0, \qquad \dot{q}_0 = -p_0.$$

Definition

A curve $(\dot{\hat{\gamma}}, u) \colon I \to T\widehat{Q} \times U$ *for the mechanical optimal control problem is*

1. *a* **normal extremal** *if it is an extremal with*
   - p_0 *being a nonzero constant;*
   - *or* $q_0 = -1$*, then* $p_0 = 0$*;*
2. *an* **abnormal extremal** *if it is an extremal with* $p_0 = q_0 = 0$.

Kinematic hamiltonian

Hamiltonian function $H_k \colon T^*\widehat{Q} \times V \to \mathbb{R}$,

$$H_k = a_0 \mathcal{G} + \sum_{s=1}^{m} a_i w^s Y_s^i \; ,$$

and Hamilton's equations

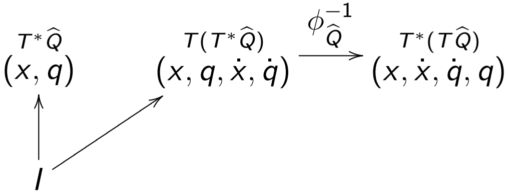
$$\begin{aligned} \dot{x}^0 &= \mathcal{G} & \dot{a}_0 &= 0 \\ \dot{x}^i &= w^s Y_s^i & \dot{a}_i &= -a_0 \frac{\partial \mathcal{G}}{\partial x^i} - a_j w^s \frac{\partial Y_s^j}{\partial x^i} \end{aligned}$$

Tulczyjew diffeomorphism

Using Tulczyjew diffeomorphism $\phi_{\widehat{Q}}$,

$$\begin{array}{ccccc} T^*(T\widehat{Q}) & \xrightarrow{\phi_{\widehat{Q}}} & T(T^*\widehat{Q}) & \xrightarrow{\tau_{T^*\widehat{Q}}} & T^*\widehat{Q} \\ (x, v, p, q) & \longmapsto & (x, q, v, p) & \longmapsto & (x, q) \end{array}$$

and in the other way round



Nonholonomic versus kinematic hamiltonian problem

$\widehat{\Lambda}\colon I \to T^*(T\widehat{Q})$ is a momenta along an optimal solution for the nonholonomic mechanical system.

Proposition

If there exists a $t_0 \in I$ such that $\langle \widehat{q}(t_0), \widehat{v}_k(t_0) \rangle \leq 0$ for every elementary perturbation vector of the kinematic system, then $\widehat{q}(t_0)$ is the initial condition for the momenta to be an extremal for the kinematic PMP.

Nonholonomic versus kinematic hamiltonian problem

Nonholonomic	Kinematic
$\dot{p}_0 = 0$	$\dot{a}_0 = 0$
$\dot{q}_0 = -p_0$	

Corollary

The abnormal optimal curves for nonholonomic mechanical system with momenta satisfying the previous hypothesis are abnormal optimal curves for the kinematic system.

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Time-optimal kinematic problem [Liu, Sussmann, 1996]

$Q = \mathbb{R}^3$. Local coordinates (x, y, z).

$D = \ker(x^2 dy - (1-x)dz) = \langle \partial/\partial x, (1-x)\partial/\partial y + x^2 \partial/\partial z \rangle$.

$g = dx \otimes dx + \psi(x)(dy \otimes dy + dz \otimes dz)$, $\psi(x) = 1/((1-x)^2 + x^4)$.

End-point conditions: $\gamma(0) = (0,0,0)$, $\gamma(t_f) = (0,1,0)$.

Kinematic hamiltonian function:

$$H_k(\widehat{a}, w_1, w_2) = a_1 w_1 + a_2 w_2 (1-x) + a_3 x^2 w_2 + a_0.$$

Local strict abnormal minimizer for the time-optimal problem:

$$(\gamma, w)\colon [0,1] \to Q \times V, \quad t \mapsto (0, t, 0, 0, 1).$$

Momenta $\widehat{a}$: $t \mapsto (0,0,0,a_3)$, $a_3 \neq 0$, along $\widehat{\gamma}$.

Nonholonomic mechanical system

Mechanical hamiltonian function: $H_m(\widehat{\Lambda}, u_1, u_2) = p_0 v_0 + q_0$

$$+p_1 v_1 + p_2 v_2 + p_3 v_3 + q_1(-\Gamma^1_{22} v_2^2 - \Gamma^1_{33} v_3^2 + u_1) + q_2 u_2(1-x) + q_3 x^2 u_2$$

Hamilton's equations:

$$\begin{aligned}
\dot{x}_0 &= v_0 & \dot{p}_0 &= 0\\
\dot{x} &= v_1 & \dot{p}_1 &= \frac{\partial \Gamma^1_{22}}{\partial x} q_1 v_2^2 + \frac{\partial \Gamma^1_{33}}{\partial x} q_1 v_3^2 + q_2 u_2 - 2x u_2 q_3\\
\dot{y} &= v_2 & \dot{p}_2 &= 0\\
\dot{z} &= v_3 & \dot{p}_3 &= 0\\
\dot{v}_0 &= 1 & \dot{q}_0 &= -p_0\\
\dot{v}_1 &= -\Gamma^1_{22} v_2^2 - \Gamma^1_{33} v_3^2 + u_1 & \dot{q}_1 &= -p_1\\
\dot{v}_2 &= u_2(1-x) & \dot{q}_2 &= -p_2 + 2 q_1 \Gamma^1_{22} v_2\\
\dot{v}_3 &= x^2 u_2 & \dot{q}_3 &= -p_3 + 2 v_3 \Gamma^1_{33} q_1
\end{aligned}$$

Mechanical extremals

The strict abnormal minimizer for the kinematic system becomes the extremal

$$\dot{\widehat{\gamma}}(t) = (t, 0, t, 0, 1, 0, 1, 0)$$

for the mechanical system.

From Hamilton's equations: $u_1 = u_2 = 1$.

Abnormal momenta, $q_0 = p_0 = 0$:

$$\widehat{\Lambda}(t) = (0, 0, 0, p_3, 0, 0, 0, -p_3 t + A).$$

Observe that $H_m(\widehat{\Lambda}(t), u_1, u_2) = 0$.

Any normal lift?

- $p_0 = -1$, then

$$\widehat{\Lambda}_1(t) = (-1, 0, 0, p_3, t + B, 0, 0, -p_3 t + A)$$

- $p_0 = 0$, $q_0 = -1$,

$$\widehat{\Lambda}_2(t) = (0, 0, 0, p_3, -1, 0, 0, -p_3 t + A).$$

Observe that

$$\begin{aligned} H_m(\widehat{\Lambda}_1(t), u_1, u_2) &= -1 + t + B, \\ H_m(\widehat{\Lambda}_2(t), u_1, u_2) &= -1 \end{aligned}$$

Contradiction with PMP!!!!!!!!

THUS, $\dot{\hat{\gamma}}$ is a strict abnormal extremal!

References

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