Reduction of Almost Poisson Brackets

Luis García-Naranjo

EPFL, Switzerland

In collaboration with

Simon Hochgerner (EPFL).

Outline

1. Motivating Example: Chapligyn's Sphere.
2. Review of Almost Poisson Brackets
3. Affine Almost Poisson Brackets
4. Complete Reduction and Hamiltonization of the Chaplygin Sphere
5. Conclusions and Current Work

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2. Review of Almost Poisson Brackets
3. Affine Almost Poisson Brackets
4. Complete Reduction and Hamiltonization of the Chaplygin Sphere
5. Conclusions and Current Work

- Nonholonomic Mechanical Systems preserve energy but are not Hamiltonian.

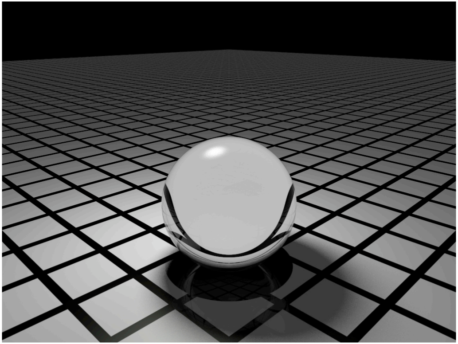
- The equations of motion can be written with respect to a bracket of functions that fails to satisfy the Jacobi identity.

  An Almost Poisson Bracket.

- Van der Schaft, Maschke, Marle, Ibort, Cantrijn, De León, Marrero, Martín de Diego, Koon, Marsden, and others.

Motivating example: Chaplygin Sphere

A sphere whose center of mass coincides with its geometric center that rolls without slipping on the plane.



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- Borisov and Mamaev [2002] *Chaplygin's Ball Rolling Problem is Hamiltonian*. After a time reparametrization write the reduced equations of motion with respect to a nonlinear bracket of functions that satisfies the Jacobi identity.
- Ehlers, Koiller, Montgomery, Rios [2004] conjecture that the system is not Hamiltonizable after reduction.

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Nonholonomic Mechanical System

- Configuration space Q, a smooth n dimensional manifold.
- (Hyper-regular) Lagrangian $\mathcal{L} : TQ \to \mathbb{R}$.
- A non-integrable constraint distribution $\mathcal{D} \subset TQ$ defined by $k < n$ constraints that are linear and homogeneous in the velocities:

$$\sum_{s=1}^{n} \beta^i_s(q)\dot{q}^s = 0, \qquad i = 1, \dots, k.$$

$\mathcal{D}_q \subset T_qQ$ is the annihilator of the one-forms on Q:

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- Lagrange-D'Alembert principle (Newton's Law):

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^s}\right) - \frac{\partial \mathcal{L}}{\partial q^s} = \underbrace{\lambda_1\beta_s^1(q) + \dots + \lambda_k\beta_s^k(q)}_{\text{Reaction Forces}}.$$

Hamiltonian Formalism

- Generalized momenta $p_s = \frac{\partial \mathcal{L}}{\partial \dot{q}^s}$. Legendre transform: $\text{Leg} : TQ \to T^*Q$.
- Hamiltonian $\mathcal{H} : T^*Q \to \mathbb{R}$.
- Constraint submanifold $\mathcal{M} = \text{Leg}(\mathcal{D}) \subset T^*Q$,

$$\mathcal{M} = \left\{ (p, q) \ : \ \sum_{s=1}^{n} \beta_s^i(q) \, \frac{\partial \mathcal{H}}{\partial p_s} = 0, \quad i = 1, \dots, k \right\}.$$

- Equations of motion

$$\dot{q}^s = \frac{\partial \mathcal{H}}{\partial p_s}, \qquad \dot{p}_s = -\frac{\partial \mathcal{H}}{\partial q^s} + \sum_{i=1}^{k} \lambda_i \beta_s^i(q).$$

Intrinsic formulation

- Ω_Q the canonical symplectic form on T^*Q.
- Equations of motion

$$\mathbf{i}_{X_{\mathrm{nh}}}\Omega_Q = d\mathcal{H} + \sum_{i=1}^{k} \lambda_i \tau^* \beta^i .$$

  X_{nh} is the nonholonomic vector field, $\tau : T^*Q \to Q$ the canonical projection.
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$$X_{\mathrm{nh}}(m) \in H_m := T_m\mathcal{M} \cap \mathrm{ann}\{\tau^*\beta^i(m), \ \ i = 1, \dots, k\}.$$

Definition of the almost Poisson Bracket in $\mathcal{M}$ (Ibort, de León, Martín de Diego, Marrero (1998))

Theorem (Weber (1986), Bates, Śniatycki (1993))

For all $m \in \mathcal{M}$ we have the symplectic decomposition

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- $X_{\mathrm{nh}}(m) = \mathcal{P}_m X_{\mathcal{H}}(m)$.
- For $f_1, f_2 \in C^\infty(\mathcal{M})$ define the nonholonomic bracket

$$\{f_1, f_2\}_{\mathrm{nh}}(m) = \Omega_Q(\mathcal{P}_m X_{f_1}(m), \mathcal{P}_m X_{f_2}(m)).$$

Properties of the Nonholonomic Bracket in $\mathcal{M}$

- Equations of motion can be written with respect to the nonholonomic bracket

$$X_{\mathrm{nh}}(f)(m) = \{f, \mathcal{H}_{\mathcal{M}}\}_{\mathrm{nh}}(m), \qquad \mathcal{H}_{\mathcal{M}} = \mathcal{H}\big|_{\mathcal{M}}.$$

- Cantrijn, de León, Martín De Diego (1999) show that the bracket $\{\cdot,\cdot\}_{\mathrm{nh}}$ so defined equals that of Van der Schaft, Maschke (1992) and Marle (1998).
- Jacobi identity is satisfied if and only if the constraints are holonomic.
- This formulation avoids dealing with Lagrange multipliers. The constraint forces are encoded in the bracket.

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- The form Ω_0 is semi-basic. That is, if v is a tangent vector to T^*Q such that $\tau_* v = 0$, then $\mathbf{i}_v \Omega_0 = 0$.

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Same relevant properties as Ω_Q!

Definition of Affine Almost Poisson Brackets

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- For $f_1, f_2 \in C^\infty(\mathcal{M})$ define the affine bracket:

$$\{f_1, f_2\tilde{\}}_{\mathrm{nh}}(m) = \tilde{\Omega}_Q(\tilde{\mathcal{P}}_m \tilde{X}_{f_1}(m), \tilde{\mathcal{P}}_m \tilde{X}_{f_2}(m)),$$

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- In general $\widetilde{\{f_1, f_2\}}_{\mathrm{nh}} \neq \{f_1, f_2\}_{\mathrm{nh}}$. Different way of encoding the constraint forces!

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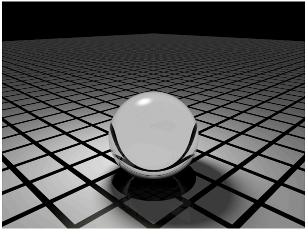

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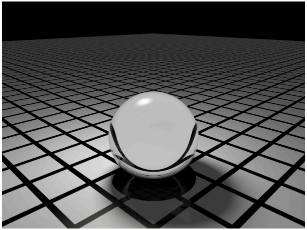
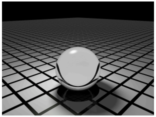


- Configuration space is $Q = SO(3) \times \mathbb{R}^2$.
- Kinetic energy Lagrangian.
- Two rolling constraints $\dot{x} = r\omega_2, \ \dot{y} = -r\omega_1$, define 8 dimensional constraint submanifold $\mathcal{M} \subset T^*Q$.

Symmetries of Chaplygin Sphere



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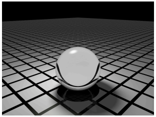

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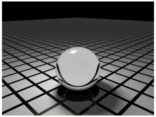

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- Conserved quantity: angular momentum about the z axis, L_z.
- The nonholonomic bracket $\{\cdot,\cdot\}_{\mathrm{nh}}$ drops to a bracket $\{\cdot,\cdot\}_{\mathcal{R}}$ on the 5 dimensional reduced space $\mathcal{R} := \mathcal{M}/SE(2) \cong S^2 \times \mathbb{R}^3$.

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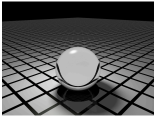

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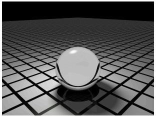

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- Key point: The bracket $\{\cdot,\cdot\}_{\mathcal{R}}$ does not admit a foliation by even dimensional leaves.

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- Equations of motion can be written with respect to the corresponding affine bracket

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Properties of the Affine Bracket

Theorem

- *The affine bracket* $\{\cdot,\cdot\tilde{\}}_{\mathrm{nh}}$ *drops to a bracket* $\{\cdot,\cdot\tilde{\}}_{\mathcal{R}}$ *on* $\mathcal{R} = \mathcal{M}/SE(2) \cong S^2 \times \mathbb{R}^3$.

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- *The bracket* $\{\cdot,\cdot\tilde{\}}_{\mathcal{R}}$ *restricts to the level sets of* L_z *that form a 4-dimensional foliation of* $\mathcal{R}$ *whose leaves are diffeomorphic to* T^*S^2.

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- *The bracket* $\{\cdot,\cdot\tilde{\}}_{\mathcal{R}}$ *restricts to the level sets of* L_z *that form a 4-dimensional foliation of* $\mathcal{R}$ *whose leaves are diffeomorphic to* T^*S^2.
- *There exists a strictly positive function* $\mu : \mathcal{R} \to \mathbb{R}$ *that "Hamiltonizes" the problem. That is, the bracket on* $\mathcal{R}$ *defined by* $\{f_1, f_2\tilde{\}}^{\mu}_{\mathcal{R}} := \mu\{f_1, f_2\tilde{\}}_{\mathcal{R}}$ *satisfies the Jacobi identity.*

Properties of the Affine Bracket

Theorem

- *The affine bracket* $\{\cdot,\cdot\tilde{\}}_{\mathrm{nh}}$ *drops to a bracket* $\{\cdot,\cdot\tilde{\}}_{\mathcal{R}}$ *on* $\mathcal{R} = \mathcal{M}/SE(2) \cong S^2 \times \mathbb{R}^3$.
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- *In the correct coordinates, the bracket* $\{\cdot,\cdot\tilde{\}}^{\mu}_{\mathcal{R}}$ *equals that given by Borisov and Mamaev (2002, 2005).*

Link with standard equations for Chaplygin's Sphere

Equations of motion:

$$\dot{K} = K \times \omega, \qquad \dot{\gamma} = \gamma \times \omega.$$

- $\omega \in \mathbb{R}^3$ is the angular velocity written in body coordinates.
  $\gamma \in \mathbb{R}^3$ is the vertical unit vector written in body coordinates.
  $K \in \mathbb{R}^3$ is the angular momentum about the contact point written in body coordinates, ($K = \mathbb{I}\omega + mr^2(\gamma \times (\omega \times \gamma))$).

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- Symmetry integral L_z equals $L_z = \langle K, \gamma \rangle$.

Link with standard equations for Chaplygin's Sphere

- Hamiltonizing function, $\mu : \mathcal{R} \to \mathbb{R}$, is given by

$$\mu(\gamma) = \sqrt{1 - mr^2 \langle (\mathbb{I} + mr^2)^{-1}\gamma, \gamma \rangle}.$$

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$$\frac{df}{dt} = \{f, \mathcal{H}_{\mathcal{R}}\tilde{\}}_{\mathcal{R}} \quad \Longleftrightarrow \quad \frac{df}{d\tau} = \mu(\gamma)\{f, \mathcal{H}_{\mathcal{R}}\tilde{\}}_{\mathcal{R}} = \{f, \mathcal{H}_{\mathcal{R}}\tilde{\}}^{\mu}_{\mathcal{R}}.$$

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- 2-degree of freedom Hamiltonian system on the level sets of L_z (together with $||\gamma|| = 1$). The function $\langle K, K\rangle$ is in involution with the Hamiltonian so, in the new time τ, we have integrability in the sense of Liouville.

Outline

Conclusions.

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Current work.

- We have shown existence of affine symplectic structures for the complete reduction of generalized Chaplygin systems with internal symmetries in higher dimensional cases.
- Hamiltonization of other higher dimensional systems with our approach is still open. Important progress done by Fedorov, Jovanovic (2004).