Symmetry analysis of nonholonomic field theories

7th AIMS International Conference
on Dynamical Systems, Differential Equations and Applications

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Monday, May 19, 2008

Outline of the talk

References

- Based on joint work with David Martín de Diego.
- Main publications:
  - J. Vankerschaver and D. Martín de Diego, *Symmetry aspects of nonholonomic field theories*, J. Phys. A: Math. Theor. **47** (2008), no. 4, 35401.
  - J. Vankerschaver, *A connection-theoretic approach to the analysis of symmetries of nonholonomic field theories*, to appear in Proceedings of the 10th conference on differential geometry and its applications, 2008.
- See also
  - J. Vankerschaver, *The momentum map for nonholonomic field theories with symmetry*, Int. J. Geom. Meth. Mod. Phys. **2** (2005), no. 6, 1029–1041, `math-ph/0507059`.

Aim of this talk

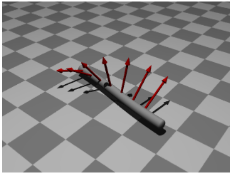
- Noether's theorem for classical field theories: symmetries give rise to conserved currents.
- Nonholonomic constraints: this picture is no longer valid.

How about nonholonomic field theories?

- Noether theorem is replaced by a *nonholonomic momentum equation*. For mechanics: Bloch *et al.* (1995). Here, we prove it for field theories.
- Interesting aside: sometimes, the concept of *infinitesimal symmetry generator* needs to be generalized...

Review: nonholonomic field theories

- The aim: to describe classical field theories whose "generalized velocities" are constrained.
- Example: nonholonomic Cosserat rod.



- Geometric treatment: jet bundles, Poincaré-Cartan forms, etc.

$$\cdots \longrightarrow J^1\pi \xrightarrow{\pi_{1,0}} Y \xrightarrow{\pi} X.$$

Nonholonomic field theories

Geometric data

Nonholonomic field theories are specified by the following elements.

- A Lagrangian $L : J^1\pi \to \mathbb{R}$;
- a constraint submanifold $\mathcal{C} \hookrightarrow J^1\pi$;
- a bundle of reaction forces $F \subset \bigwedge^{n+1}(J^1\pi)$, whose sections are
  1. 1-contact, implying that $(j^1\phi)^*\Phi = 0$;
  2. n-horizontal, *i.e.* $i_V i_W \Phi = 0$ for any two $V, W \in V\pi_1$.

(there are some rank conditions to be fulfilled as well.)

- Starting point for constrained De Donder-Weyl equation, nonholonomic projector, etc.
- Treated by different groups: Krupková *et al.*; de León & D. Martín de Diego; Cantrijn & JV.

Nonholonomic field equations

Theorem

A field ϕ such that $j^1\phi \subset \mathcal{C}$ is an extremum of the action integral under admissible variations if and only if it satisfies the following nonholonomic Euler-Lagrange equations:

$$(j^1\phi)^* i_W \Omega_L = 0$$

for all vector fields W such that $(j^1\phi)^*(i_W \Phi) = 0$, for $\Phi \in F$.

- Derived by varying the action integral $S = \int L \mathrm{d}^n x$ under *admissible* variations.
- In coordinates:

$$\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial L}{\partial y^a_\mu}\right) - \frac{\partial L}{\partial y^a} = \lambda_\alpha A^\alpha_\mu, \quad \text{and} \quad \varphi^\alpha(j^1\phi(x)) = 0,$$

with $\lambda_\alpha A^\alpha_\mu \in F$.

Nonholonomic symmetries

Let G be a Lie group acting on Y and by prolongation also on $J^1\pi$.

$$\boxed{\text{Assume } G \text{ leaves invariant } L, \mathcal{C} \text{ and } F.}$$

The bundle $\mathfrak{g}^{\mathcal{E}}$ over Y (Admissible symmetry generators)

Denote by $\mathfrak{g}^{\mathcal{E}}(y)$ the linear subspace of $\mathfrak{g}$ consisting of all $\xi \in \mathfrak{g}$ such that

$$j^1\xi_Y(\gamma) \lrcorner F = 0 \quad \text{for all } \gamma \in \mathcal{C} \cap \pi_{1,0}^{-1}(y).$$

$\Rightarrow$ vector bundle $\mathfrak{g}^{\mathcal{E}}$ over Y.

To any section $\bar{\xi}$ of $\mathfrak{g}^{\mathcal{E}}$, one can associate a vector field $\tilde{\xi}_Y$ on Y by putting

$$\tilde{\xi}_Y(y) = \left[\bar{\xi}(y)\right]_Y (y).$$

Nonholonomic momentum map

Definition

A nonholonomic symmetry is a section $\bar{\xi}$ of $\mathfrak{g}^{\mathcal{E}}$ such that the associated vector field $\tilde{\xi}_Y$ is π-projectable; *i.e.* there exists a vector field $\tilde{\xi}_X$ on X such that $T\pi \circ \tilde{\xi}_Y = \tilde{\xi}_X \circ \pi$.

Remark: the projectability condition is necessary!

Definition

The nonholonomic momentum map is the map $J^{\mathrm{n.h.}} : \mathcal{C} \to \bigwedge^n(J^1\pi) \otimes g^{\mathcal{E}}$ constructed as follows. Let $\bar{\xi}$ be a section of $\mathfrak{g}^{\mathcal{E}}$; define the component $J^{\mathrm{n.h.}}_{\bar{\xi}}$ as

$$J^{\mathrm{n.h.}}_{\bar{\xi}} = i_{j^1\tilde{\xi}_Y}\Theta_L.$$

Then $\left\langle J^{\mathrm{n.h.}}(\gamma), \bar{\xi}(\gamma)\right\rangle = J^{\mathrm{n.h.}}_{\bar{\xi}}(\gamma)$.

Nonholonomic momentum equation

- Classical Noether theorem: symmetry $\Rightarrow$ conservation law.
- Nonholonomic constraints: no longer true; nonholonomic momentum equation.

Theorem (nonholonomic momentum equation)

If ϕ is a solution of the nonholonomic field equations, then for any nonholonomic symmetry $\bar{\xi}$ the momentum map $J^{\mathrm{n.h.}}_{\bar{\xi}}$ satisfies

$$(j^1\phi)^*(\mathrm{d}J^{\mathrm{n.h.}}_{\bar{\xi}}) = (j^1\phi)^*\left(\mathscr{L}_{j^1\tilde{\xi}_Y}(L\eta)\right).$$

- "Normal" Noether theorem with a nonzero RHS
- Extension of a result by Bloch *et al.* and Cantrijn *et al.*

Sketch of proof

- Start from the identity

$$\mathrm{d}J_{\bar{\xi}}^{\mathrm{n.h.}} = \mathscr{L}_{j^1\tilde{\xi}_Y}\Theta_L + i_{j^1\tilde{\xi}_Y}\Omega_L.$$

- Take the pullback along a solution ϕ of the nonholonomic EL equations:

$$(j^1\phi)^*(\mathrm{d}J_{\bar{\xi}}^{\mathrm{n.h.}}) = (j^1\phi)^*\mathscr{L}_{j^1\tilde{\xi}_Y}\Theta_L.$$

- Now, note that

$$\mathscr{L}_{j^1\tilde{\xi}_Y}\Theta_L = \underbrace{\mathscr{L}_{j^1\tilde{\xi}_Y}\left(S_\eta^*(\mathrm{d}L)\right)}_{\text{contact}} + \mathscr{L}_{j^1\tilde{\xi}_Y}(L\eta).$$

- Substitute back and conclude that the nonholonomic momentum lemma holds.

Generalized vector fields

Definition

A *vector field along the projection* $\pi_{1,0}$ is a map $X : J^1\pi \to TY$ such that

$$X(\gamma) \in T_{\pi_{1,0}(\gamma)} Y.$$

Let G be a Lie group acting vertically on Y and leaving invariant $(L, \mathcal{C}, F)$.

Definition

A *generalized nonholonomic symmetry* is a section $\bar{\xi}$ of the pull-back bundle $\pi_{1,0}^* \mathfrak{g}^{\mathcal{E}}$.

To any generalized symmetry, one can associate a vertical vector field $\tilde{\xi}_Y$ along $\pi_{1,0}$ by putting

$$\tilde{\xi}_Y(\gamma) = \left[\bar{\xi}(\gamma)\right]_Y (y) \qquad \text{where } y = \pi_{1,0}(\gamma).$$

Momentum equation: extension

Theorem

If ϕ is a solution of the nonholonomic field equations, then for any generalized nonholonomic symmetry $\bar{\xi}$ the momentum map $J_{\bar{\xi}}^{\mathrm{n.h.}}$ satisfies

$$(j^2\phi)^*(\mathrm{d}J_{\bar{\xi}}^{\mathrm{n.h.}}) = (j^2\phi)^* \left(\mathscr{L}_{j^1\tilde{\xi}_Y}(L\eta) \right).$$

Proof: similar to first case but more involved.

Example: the Benenti system

Describes two particles, whose velocities are constrained to be parallel.

- Configuration space $Q := \mathbb{R}^2 \times \mathbb{R}^2$; put $X = \mathbb{R}$ and $Y = \mathbb{R} \times Q$.
- Kinetic energy Lagrangian:

$$L = \frac{m}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m}{2}(\dot{x}_2^2 + \dot{y}_2^2).$$

- Nonlinear constraint: $\varphi \equiv \dot{x}_1 \dot{y}_2 - \dot{x}_2 \dot{y}_1 = 0$. The bundle of reaction forces is spanned by

$$\Phi = \dot{y}_2 \theta_{x_1} + \dot{x}_1 \theta_{y_2} - \dot{y}_1 \theta_{x_2} - \dot{x}_2 \theta_{y_1},$$

where $\theta_{q^a} = \mathrm{d}q^a - \dot{q}^a \mathrm{d}t$.

Benenti system: nonholonomic symmetries

Symmetry action

The group $\mathbb{R}^2 \times \mathbb{R}^2$ acts vertically on Y by translations. Infinitesimal generator:

$$\xi_Y = \alpha^1 \frac{\partial}{\partial x^1} + \alpha^2 \frac{\partial}{\partial x^2} + \beta^1 \frac{\partial}{\partial y^1} + \beta^2 \frac{\partial}{\partial y^2}.$$

- Action leaves invariant L, $\mathcal{C}$, and F.
- Nonholonomic symmetries: sections $(\alpha \dot{x}_1, \beta \dot{y}_1, \gamma \dot{x}_2, \delta \dot{y}_2)$ of $\pi_{1,0}^* \mathfrak{g}^{\mathcal{E}}$ (not all nonholonomic symmetries are of this form). Associated vector field annihilates F along $\mathcal{C}$:

$$j^1 \tilde{\xi}_Y \lrcorner\, \Phi = (\alpha - \beta - \gamma + \delta)(\dot{x}_1 \dot{y}_2 - \dot{x}_2 \dot{y}_1) \approx 0.$$

Benenti system: conservation laws

The nonholonomic Noether theorem yields the following conservation laws. Schematically:

$(\alpha, \beta, \gamma, \delta)$	
(1, 1, 0, 0)	$\dot{x}_1\ddot{x}_1 + \dot{y}_1\ddot{y}_1 = 0$
(0, 0, 1, 1)	$\dot{x}_2\ddot{x}_2 + \dot{y}_2\ddot{y}_2 = 0$
(1, 0, 0, -1)	$\dot{x}_1\ddot{x}_1 - \dot{y}_2\ddot{y}_2 = 0$

Lesson to be learned

For nonlinear constraints, nonholonomic symmetries will generally be vector fields along the projection.

Example: nonholonomic Cosserat rod

Describes a flexible rod, rolling without sliding on a horizontal plane.

- Configuration bundle $\pi : Y \to X$, where $X = [0, \ell] \times \mathbb{R}$ (space and time) and $Y = X \times \mathbb{R}^2 \times \mathbb{S}^1$, with bundle coordinates $(s, t; x, y, \theta)$.
- Lagrangian:

$$L = \frac{\rho}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\alpha}{2}\dot{\theta}^2 - \frac{1}{2}\left(\beta(\theta')^2 + K\kappa^2\right)$$

where $\kappa^2 = (x'')^2 + (y'')^2$.

- The constraints are given by

$$\dot{x} + R\dot{\theta}y' = 0 \quad \text{and} \quad \dot{y} - R\dot{\theta}x' = 0.$$

Here, ρ, α, β, K, and R are real parameters.

Nonholonomic symmetries: time translation

- Translations in time leave invariant $(L, \mathcal{C}, F)$.
- Infinitesimal generator $\xi_Y = \frac{\partial}{\partial t}$: horizontal nonholonomic symmetry.

Components of the nonholonomic momentum map:

$$(j^3\phi)^* J_1^{\mathrm{n.h.}} = \mathcal{E}\mathrm{d}s + \left[-Kx'''\dot{x} + -Ky'''\dot{y} + \beta\theta'\dot{\theta} + K(x''\dot{x}' + y''\dot{y}') \right] \mathrm{d}t,$$

where the energy density is given by

$$\mathcal{E} = \frac{\rho}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\alpha}{2}\dot{\theta}^2 + \frac{K}{2}((x'')^2 + (y'')^2) + \frac{\beta}{2}(\theta')^2.$$

Nonholonomic momentum equations $\Rightarrow$ conservation of energy

Genuine conservation law:

$$(j^3\phi)^* \mathrm{d} J_1^{\mathrm{n.h.}} = 0$$

(RHS vanishes as $\frac{\partial}{\partial t}$ is a constant section.)

Cosserat rod: momentum conservation

- $\mathbb{R}^2 \times \mathbb{S}^1$ acts vertically on Y by spatial translations;
- Nonholonomic symmetry:

$$\tilde{\xi} = -Ry' \frac{\partial}{\partial x} + Rx' \frac{\partial}{\partial y} + \frac{\partial}{\partial \theta}.$$

- Nonholonomic momentum map:

$$J^{\mathrm{n.h.}}_{\tilde{\xi}} = -\left[\rho(Rx'\dot{y} - Ry'\dot{x}) + \alpha\dot{\theta}\right]\mathrm{d}s - \left[KR(y'x''' - x'y''') + \beta\theta'\right]\mathrm{d}t,$$

- Modified conservation law:

$$Ry'(\rho\ddot{x} + Kx'''') - Rx'(\rho\ddot{y} + Ky'''') = \alpha\ddot{\theta} - \beta\theta''.$$

Conclusions & Outlook

1. Nonholonomic symmetries come in (at least) two different varieties;
2. Each of them gives rise to a non-trivial “conservation” equation.

Outlook

1. Will this setup detect other symmetries in mechanics with nonlinear nonholonomic constraints?
2. *Can these modified conservation laws be used to help us solve the system in any way?*

Thank you for your time!

References

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