

Geometry of variational nonholonomic Lagrangian systems with symmetries

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7th AIMS International Conference on
Dyn. Systems, Diff. Equations and Applications
Nonholonomic constraints in Mechanics and Optimal Control Theory
Arlington, Texas, USA, May 18 - 21, 2008

arXiv:0706.2789: Singular lagrangian systems and variational constrained
mechanics on Lie algebroids

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Constrained variational calculus has a rich geometric structure

Arnold (1998)

Our motivation:

- Optimal control problems of nonholonomic systems with symmetries
- Reduction of subriemannian problems

Bloch (2003)

Cortés, de León, Martín de Diego, Martínez (2002,2004)

Jurdjevic, Sharpe (1998)

Koon, Marsden (1997)

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PLAIN OF THE TALK

- 1 Standard variational nonholonomic (vakonomic) Lagrangian systems
- 2 Lie algebroids
- 3 Variational nonholonomic (vakonomic) Lagrangian systems on a Lie algebroid
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 - 4.2 Atiyah algebroids and reduction in subriemannian geometry
 - 4.3 Optimal control systems on Lie algebroids

1. Variational nonholonomic (vakonomic) Lagrangian systems

$Q \equiv$ **The configuration manifold**

$\tau_{TQ} : TQ \rightarrow Q$ the canonical projection

$L : TQ \rightarrow \mathbb{R}$ **the Lagrangian function**

$M \hookrightarrow TQ$ **the constraint submanifold**

$\tau_M = (\tau_{TQ})|_M : M \rightarrow Q$ is a fibration

$q_0, q_1 \in Q$

$$\tilde{\mathcal{C}}(q_0, q_1) = \left\{ \begin{array}{l} a : [t_0, t_1] \rightarrow Q \in \mathcal{C}^2(q^1, q^2) \quad / \quad \dot{a}(t) \in M_{a(t)} = M \cap \tau_Q^{-1}(a(t)), \forall t \\ a(t_0) = q_0 \text{ and } a(t_1) = q_1 \end{array} \right\}$$

The action functional

$$\tilde{\delta S} : \tilde{\mathcal{C}}(q_0, q_1) \rightarrow \mathbb{R}, \quad a \rightarrow \tilde{\delta S}(a) = \int_{t_0}^{t_1} L(\dot{a}(t)) dt$$

The infinitesimal variations

$a \in \tilde{\mathcal{C}}(q_0, q_1) \implies a_s = \phi_s(a), \quad \forall s$

$\phi_s : Q \rightarrow Q$ the flow of a vector field X on Q

$$X(q_0) = 0, \quad X(q_1) = 0, \quad (T\phi_s)(M) = M$$

Remark: $(T\phi_s)(M) = M \Leftrightarrow X|_M^c$ is tangent to M

$X^c \equiv$ the standard complete lift of X to TQ

Problem

To find critical points of the action functional $\widetilde{\delta S}$

Answer

The (normal) solutions of the constrained (vakonomic) problem

Reduction of standard variational nonholonomic (vakonomic) Lagrangian systems

$$L : TQ \rightarrow \mathbb{R}, \quad M \hookrightarrow TQ$$

$$(L, M) \text{ a constrained system, } \quad \tilde{L} = L|_M$$

$p : Q \rightarrow Q/G$ a principal G -bundle

$\Phi : G \times Q \rightarrow Q$ the free action of G on Q

The action Φ **preserves** the vakonomic system

$$(T\Phi_g)(M) \subseteq M, \quad L \circ (T\Phi_g)|_M = L$$

A reduced constrained system

$L|G : TQ|G \rightarrow \mathbb{R}$ the reduced Lagrangian

M/G the reduced constrained submanifold of TQ/G

$\widetilde{L|G} : M/G \rightarrow \mathbb{R}$ the reduced constrained Lagrangian

The solutions of the variational constrained system (L, M) project, via p , on the solutions of the reduced constrained system

Problem

$(L|G, M/G)$ is not an standard vakonomic Lagrangian system (TQ/G is not, in general, a tangent bundle)

However, TQ/G admits a Lie algebroid structure

2.- Lie algebroids

$\tau_E : E \rightarrow Q$ a real vector bundle over Q

$\Gamma(E) \equiv$ sections of $\tau_E : E \rightarrow Q$

A Lie algebroid structure $([\cdot, \cdot]_E, \rho_E)$ on E

(i) $[\cdot, \cdot]_E : \Gamma(E) \times \Gamma(E) \rightarrow \Gamma(E)$ a Lie bracket on $\Gamma(E)$

(ii) $\rho_E : E \rightarrow TQ$ a vector bundle morphism such that

$$[X, fY]_E = f[X, Y]_E + \rho_E(X)(f)Y, \quad X, Y \in \Gamma(E), f \in C^\infty(Q)$$

$\rho_E \equiv$ the anchor map

(i) and (ii) $\implies \rho_E : \Gamma(E) \rightarrow \mathfrak{X}(Q)$ is a Lie algebra morphism

K.C. H. Mackenzie (2005)

The local structure equations

$(q^A) \equiv$ local coordinates on Q
 $\{e_i\}$ a local basis of $\Gamma(E)$



$(q^A, v^i) \equiv$ the local coordinates on E

The local structure functions $C_{\alpha\beta}^\gamma$ and $(\rho_E)_\alpha^i$

$$\rho_E(e_\alpha) = (\rho_E)_\alpha^A \frac{\partial}{\partial q^A}, \quad \llbracket e_\alpha, e_\beta \rrbracket_E = C_{\alpha\beta}^\gamma e_\gamma$$

$$(\rho_E)_i^A \frac{\partial (\rho_E)_j^B}{\partial q^A} - (\rho_E)_j^A \frac{\partial (\rho_E)_i^B}{\partial q^A} = (\rho_E)_k^B C_{ij}^k$$

$$\sum_{\text{cyclic}(i,j,k)} [(\rho_E)_i^A \frac{\partial C_{jk}^l}{\partial q^A} + C_{in}^l C_{jk}^m] = 0$$

The local structure equations

EXAMPLES:

$p : Q \rightarrow Q/G$ a principal G -bundle

$\tau_{TQ/G} : TQ/G \rightarrow Q/G, [v_q] \rightarrow p(q)$ is a real vector bundle over Q/G

$\Gamma(TQ/G) = \{X \in \mathfrak{X}(Q) / X \text{ is } G\text{-invariant}\}$

- $X, Y \in \mathfrak{X}(Q)$ are G -invariant $\Rightarrow [X, Y]$ also is G -invariant

\Downarrow

$[\cdot, \cdot]_{TQ/G} : \Gamma(TQ/G) \times \Gamma(TQ/G) \rightarrow \Gamma(TQ/G)$ a Lie bracket on $\Gamma(TQ/G)$

- $X \in \mathfrak{X}(Q)$ is G -invariant $\Rightarrow X$ is p -projectable

\Downarrow

$\rho_{TQ/G} : TQ/G \rightarrow T(Q/G), [v_q] \rightarrow (T_q p)(v_q)$ the anchor map

$\tau_{TQ/G} : TQ/G \rightarrow Q/G$ the **Atiyah algebroid**

associated with the principal G -bundle $p : Q \rightarrow Q/G$

- $G = \{e\} \implies TQ/G = TQ$ (tangent bundles)

$\Gamma(TQ) = \mathfrak{X}(Q)$ $[[\cdot, \cdot]]_{TQ} = [\cdot, \cdot]$ the standard Lie bracket of vector fields on Q

$\rho_{TQ} : TQ \rightarrow TQ$ is the identity map

- $G = Q \implies TG/G \cong (G \times \mathfrak{g})/G \cong \mathfrak{g}$ (lie algebras)

$[[\cdot, \cdot]]_{\mathfrak{g}}$ is the Lie bracket $[\cdot, \cdot]_{\mathfrak{g}}$, $\rho_{\mathfrak{g}} : \mathfrak{g} \rightarrow \{\text{a point}\}$ is the constant map

- $p : Q = G \times N \rightarrow N$ a trivial principal G -bundle

\Downarrow

$$TQ/G \cong \frac{TG \times TN}{G} \cong \frac{G \times \mathfrak{g} \times TN}{G} \cong \mathfrak{g} \times TN$$

$$\tau_{TQ/G} : TQ/G \cong \mathfrak{g} \times TN \rightarrow Q/G = N$$

(product of a tangent bundle and a Lie algebra)

$$\Gamma(\mathfrak{g} \times TN) \cong C^\infty(N, \mathfrak{g}) \oplus \mathfrak{X}(N)$$

$$\xi, \xi' \in \mathfrak{g}, \quad X, X' \in \mathfrak{X}(N) \implies [[(\xi, X), (\xi', X')]]_{\mathfrak{g} \times TN} = ([\xi, \xi']_{\mathfrak{g}}, [X, X'])$$

$$\rho_{\mathfrak{g} \times TN}(\xi, X) = X$$

Admissible curves on a Lie algebroid

$(E, [\cdot, \cdot]_E, \rho_E)$ a Lie algebroid over Q

$\alpha : I \rightarrow E$ a curve on E

$$\alpha \text{ admissible} \iff \rho_E \circ \alpha = \frac{d}{dt}(\alpha \circ \tau_E)$$

A particular case:

$$E = TQ \quad (\rho_E = Id)$$

$\alpha : I \rightarrow TQ$ is admissible $\iff \alpha$ is the **tangent lift** of a curve $a : I \rightarrow Q$ on Q

Complete Lifts of sections in a Lie algebroid

J. Grabowski, P. Urbanski (1997)

$(E, \llbracket \cdot, \cdot \rrbracket_E, \rho_E)$ a Lie algebroid over Q

$X : Q \rightarrow E \in \Gamma(E)$

\Downarrow

$X^c \in \mathfrak{X}(E)$ the complete lift to E of X

$(q^A, v^i) \equiv$ local fibred coordinates on E

$X = X^i e_i$

$$X^c = X^i (\rho_E)_i^A \frac{\partial}{\partial q^A} + ((\rho_E)_j^A \frac{\partial X^k}{\partial q^A} - X^i C_{ij}^k) v^j \frac{\partial}{\partial v^k}$$

A particular case:

$$E = TQ \implies (\rho_E)_\beta^A = \delta_\beta^A, \quad C_{\alpha\beta}^\gamma = 0$$

$$X = X^B \frac{\partial}{\partial q^B} \in \mathfrak{X}(Q) = \Gamma(E), \quad X^c = X^B \frac{\partial}{\partial q^B} + \frac{\partial X^B}{\partial q^A} v^A \frac{\partial}{\partial v^B}$$

The standard complete lift of $X \in \mathfrak{X}(Q)$

$(E, [\cdot, \cdot]_E, \rho_E)$ a Lie algebroid over Q

$L : E \rightarrow \mathbb{R}$ a Lagrangian function

$M \hookrightarrow E$ the constraint submanifold

$\tau_M = (\tau_E)|_M : M \rightarrow Q$ is a fibration

$(q_0, q_1) \in Q$

$$\mathcal{C}(q_0, q_1) = \left\{ \alpha : [t_0, t_1] \rightarrow E \quad / \quad \begin{array}{l} \alpha \text{ is admissible, } \tau_E(\alpha(t_0)) = q_0, \\ \tau_E(\alpha(t_1)) = q_1 \text{ and } \alpha(t) \in M, \forall t \end{array} \right\}$$

The action functional

$$\delta S : \mathcal{C}(q_0, q_1) \rightarrow \mathbb{R}, \quad \alpha \rightarrow \delta S(\alpha) = \int_{t_0}^{t_1} L(\alpha(t)) dt$$

The infinitesimal variations

$$\alpha \in \mathcal{C}(q_0, q_1) \Rightarrow \alpha_s = \Psi_s(\alpha), \forall s$$

$\Psi_s : E \rightarrow E$ is the flow of X^c , with $X \in \Gamma(E)$ and

$$X(q_0) = 0, \quad X(q_1) = 0 \text{ and } \Psi_s(M) = M$$

Problem

To find the critical points of the action functional δS

The local expression

$(q^A) \equiv$ local coordinates on Q

$\{e_i\}$ a local basis of $\Gamma(E)$

$(q^A, v^i) \equiv$ the local coordinates on E

$(\rho_E)_i^A, C_{ij}^k$ the local structure functions on E

$$M = \{(q^A, v^i) = (q^A, v^a, v^\alpha) / v^\alpha = \Psi^\alpha(q^A, v^a), \forall \alpha\}$$

Vakonomic equations

$$\begin{cases} \dot{q}^A = v^a (\rho_E)_a^A + \Psi^\alpha (\rho_E)_\alpha^A, \\ \frac{d}{dt} \left(\frac{\partial \tilde{L}}{\partial v^a} - p_\alpha \frac{\partial \Psi^\alpha}{\partial v^a} \right) = \left(\frac{\partial \tilde{L}}{\partial q^A} - p_\alpha \frac{\partial \Psi^\alpha}{\partial q^A} \right) (\rho_E)_a^A - v^b C_{ab}^k p_k - \Psi^\alpha C_{a\alpha}^k p_k \\ \dot{p}_\alpha = \left(\frac{\partial \tilde{L}}{\partial q^A} - p_\beta \frac{\partial \Psi^\beta}{\partial q^A} \right) (\rho_E)_\alpha^A - v^a C_{\alpha a}^k p_k - \Psi^\beta C_{\alpha\beta}^k p_k \end{cases}$$

$$p_a = \frac{\partial \tilde{L}}{\partial v^a} - p_\alpha \frac{\partial \Psi^\alpha}{\partial v^a}, \quad \tilde{L} = L|_M : M \rightarrow \mathbb{R}$$

4.- EXAMPLES

4.1 Lie algebras and Optimization theorem for nonholonomic systems on Lie groups

The Lie algebroid E :

$\mathfrak{g} \equiv$ a real Lie algebra of finite dimension

The Lagrangian function $l : \mathfrak{g} \rightarrow \mathbb{R}$

The constraint submanifold M :

$M = \mathfrak{C}$ an affine subspace of \mathfrak{g} modelled over the vector space C

$e_0 \in \mathfrak{C}, \quad e_0 \neq 0$

$\{e_i\} = \{e_a, e_o, e_{\bar{a}}\} = \{e_a, e_{\alpha}\}$ a basis of \mathfrak{g}

$\{e_a\}$ is a basis of C

$[e_i, e_j] = C_{ij}^k e_k$

A curve $\sigma : t \rightarrow (v^a(t), v^0(t), v^{\bar{a}}(t)) = (v^a(t), 1, 0, \dots, 0)$ in \mathfrak{C}
 σ is a solution of the vakonomic equations for (L, \mathfrak{C})

\Updownarrow

$$\left\{ \begin{array}{l} \frac{d}{dt} \left(\frac{\partial \tilde{L}}{\partial v^a} \right) = - \frac{\partial \tilde{L}}{\partial v^c} (v^b C_{ab}^c + C_{a0}^c) - p_\beta (v^b C_{ab}^\beta + C_{a0}^\beta), \\ \dot{p}_\alpha = - \frac{\partial \tilde{L}}{\partial v^c} (v^b C_{\alpha b}^c + C_{\alpha 0}^c) - p_\beta (v^b C_{\alpha b}^\beta + C_{\alpha 0}^\beta). \end{array} \right.$$

$$\begin{aligned} t \rightarrow \gamma(t) &= \left(\frac{\partial \tilde{L}}{\partial v^a} \Big|_{\sigma(t)}, p_\alpha(t) \right) \\ &= \left(\frac{\partial \tilde{L}}{\partial v^a} \Big|_{\sigma(t)}, 0 \right) + \lambda(t) \text{ a curve in } \mathfrak{g}^* \end{aligned}$$

\Downarrow

γ satisfies **the Euler-Poincaré equations**

$$\frac{d}{dt} \left(\frac{\partial \tilde{L}}{\partial v} + \lambda \right) = \text{ad}_\sigma^* \left(\frac{\partial \tilde{L}}{\partial v} + \lambda \right)$$

Optimization theorem for nonholonomic systems on Lie groups

Koon, Marsden (1997)

4.2.- Atiyah algebroids and reduction in subriemannian geometry

$p : Q \rightarrow \bar{Q} = Q/G$ a principal G -bundle

D a distribution on Q

Assumption 1 $T_q Q = D_q + V_q p, \forall q \in Q$

$\langle \cdot, \cdot \rangle_D$ is a bundle metric on D

Assumption 2: D and $\langle \cdot, \cdot \rangle_D$ are G -invariant

$$q \in Q \Rightarrow S_q = D_q \cap V_q(p)$$

The nonholonomic connection

$q \in Q \Rightarrow H_q^{nh} =$ the orthogonal complement S_q^\perp of S_q in D_q

Bloch, Krishnaprasad, Marsden and Murray (1996)

Lagrangian function on D

$$L : D \rightarrow \mathbb{R}, \quad L(v_q) = \frac{1}{2} \langle v_q, v_q \rangle_D$$

$\tau_{TQ/G} : TQ/G \rightarrow \bar{Q} = Q/G$ the Atiyah algebroid

D/G is a vector subbundle of TQ/G

$I = L/G : D/G \rightarrow \mathbb{R}$ the reduced constrained Lagrangian function

$(I, D/G)$ a vakonomic system on the Atiyah algebroid

$$\tau_{TQ/G} : TQ/G \rightarrow \bar{Q} = Q/G$$

The local expressions:

$(x^{\bar{A}})$ local coordinates on $\bar{Q} = Q/G$

$$\{e_i\} = \{e_{\bar{A}} = \left(\frac{\partial}{\partial x^{\bar{A}}}\right)^h, e_\theta\} = \{e_{\bar{A}}, e_a, e_\alpha\}$$

a local basis of $\Gamma(TQ/G) \cong \{X \in \mathfrak{X}(Q)/X \text{ is } G\text{-invariant}\}$

$\{e_{\bar{A}}, e_a\}$ a local basis of $\Gamma(D/G)$

e_θ is a vertical G -invariant vector field on Q

$(x^{\bar{A}}, \dot{x}^{\bar{A}}, v^\alpha)$ local coordinates on TQ/G

$$[[e_{\bar{A}}, e_{\bar{B}}]]_{TQ/G} = B_{\bar{A}\bar{B}}^i e_i, \quad [[e_{\bar{A}}, e_{\theta}]]_{TQ/G} = \mu_{\bar{A}\theta}^{\theta'} e_{\theta'}, \quad [[e_{\theta}, e_{\theta'}]]_{TQ/G} = C_{\theta\theta'}^{\theta''} e_{\theta''}$$

$$\rho_{TQ/G}(e_{\bar{A}}) = \frac{\partial}{\partial x^{\bar{A}}}, \quad \rho_{TQ/G}(e_{\theta}) = 0$$

$\sigma : t \rightarrow \sigma(t) = (x^{\bar{A}}(t), \dot{x}^{\bar{A}}(t), v^a(t), 0)$ a curve in D/G

σ is a solution of the vakonomic equations for the system $(I, D/G)$

\Updownarrow

$$\left\{ \begin{array}{l} \frac{d}{dt} \left(\frac{\partial I}{\partial \dot{x}^{\bar{A}}} \right) = \frac{\partial I}{\partial x^{\bar{A}}} - (\dot{x}^{\bar{B}} B_{\bar{B}\bar{B}}^a + v^b \mu_{\bar{A}b}^a) \frac{\partial I}{\partial v^a} - (\dot{x}^{\bar{B}} B_{\bar{A}\bar{B}}^{\alpha} + v^b \mu_{\bar{A}b}^{\alpha}) p_{\alpha} \\ \frac{d}{dt} \left(\frac{\partial I}{\partial v^a} \right) = (\dot{x}^{\bar{B}} \mu_{\bar{B}a}^c - v^b C_{ab}^c) \frac{\partial I}{\partial v^c} + (\dot{x}^{\bar{B}} \mu_{\bar{B}a}^{\alpha} - v^b C_{ab}^{\alpha}) p_{\alpha} \\ \dot{p}_{\alpha} = (\dot{x}^{\bar{A}} \mu_{\bar{A}\alpha}^b - v^a C_{\alpha a}^b) \frac{\partial I}{\partial v^b} + (\dot{x}^{\bar{B}} \mu_{\bar{A}\alpha}^{\beta} - v^a C_{\alpha a}^{\beta}) p_{\beta} \end{array} \right.$$

$\{e_{\bar{A}}, a_a\}$ an orthonormal local basis of $\Gamma(D/G)$

\Downarrow

$$l(x^{\bar{A}}, \dot{x}^{\bar{A}}, v^a) = \frac{1}{2}(\sum (\dot{x}^{\bar{A}})^2 + \sum (v^a)^2)$$

$$\left\{ \begin{array}{l} \ddot{x}^{\bar{A}} = -\sum (\dot{x}^{\bar{B}} B_{\bar{A}\bar{B}}^a + v^b \mu_{\bar{A}b}^a) v^a - (\dot{x}^{\bar{B}} B_{\bar{A}\bar{B}}^\alpha + v^b \mu_{\bar{A}b}^\alpha) p_\alpha, \\ \dot{v}^a = \sum (\dot{x}^{\bar{B}} \mu_{\bar{B}a}^c - v^b C_{ab}^c) v^c + (\dot{x}^{\bar{B}} \mu_{\bar{B}a}^\alpha - v^b C_{ab}^\alpha) p_\alpha, \\ \dot{p}_\alpha = \sum (\dot{x}^{\bar{A}} \mu_{\bar{A}\alpha}^b - v^a C_{\alpha a}^b) v^b + (\dot{x}^{\bar{A}} \mu_{\bar{A}\alpha}^\beta - v^a C_{\alpha a}^\beta) p_\beta \end{array} \right.$$

Application: Reduction of subriemannian problems

4.3 Optimal control systems on Lie algebroids

$(E, \llbracket \cdot, \cdot \rrbracket_E, \rho_E)$ a Lie algebroid over a manifold Q

$\pi : C \rightarrow Q$ a fibration

$\sigma : C \rightarrow E$ a section along π

$l : C \rightarrow \mathbb{R}$ an **index (cost) function**

(π, σ, l) an **optimal control system** on E

Martínez (2004)

$(q_0, q_1) \in Q$

Problem: To find the curves $t \rightarrow c(t)$ on C such that $\pi(c(t_0)) = q_0$, $\pi(c(t_1)) = q_1$ and

$$\rho_E \circ \sigma \circ c = \frac{d}{dt}(\pi \circ c)$$

and minimize $\int_{t_0}^{t_1} l(c(t))dt$

The vakonomic system associated with the optimal control system

Hypothesis: $M = \sigma(C)$ is an embedded submanifold of E



$\sigma : C \rightarrow M$ is a diffeomorphism

$L = I \circ \sigma^{-1} : M \rightarrow \mathbb{R}$ is a Lagrangian function

(L, M) is the vakonomic system on E associated
with the optimal control system

The local expressions

$(q^A) \equiv$ local coordinates on Q (the *state variables*)

$\{e_i\}$ a local basis of $\Gamma(E)$

$C_{ij}^k, (\rho_E)_i^A$ local structure functions on E

$(q^A, v^i) \equiv$ local coordinates on E

$\pi(q^A, u^a) = q^A$ ($u^a \equiv$ the control functions)

$\sigma(q^A, u^a) = (q^A, u^a, \sigma^\alpha(q^A, u^a))$

Problem:

To find the curves $t \rightarrow c(t) = (q^A(t), u^a(t))$ such that
 $q(t_0) = q_0, q(t_1) = q_1,$

$$\frac{dq^A}{dt} = (\rho_E)_a^A u^a + (\rho_E)_\alpha^A \sigma^\alpha, \quad \forall A$$

and minimize $\int_{t_0}^{t_1} l(q^A(t), u^a(t)) dt.$

A particular case: $E = TQ$

An standard optimal control system on Q

An explicit example on an Atiyah algebroid

A (homogeneous) sphere of radius $r = 1$, mass m and inertia about any axis k^2 , rolls without sliding on a horizontal table which rotates with constant angular velocity Ω about the q^3 -axis. The coordinates of the point of contact of the sphere with the plane are (q^1, q^2) .

Neimark, Fufaev (1972)

Configuration space: $Q = \mathbb{R}^2 \times SO(3)$

A trivial principal bundle $p : Q = \mathbb{R}^2 \times SO(3) \rightarrow \mathbb{R}^2$

Nonholonomic system on the Atiyah algebroid

$$E = TQ/SO(3) \cong \frac{T\mathbb{R}^2 \times SO(3) \times \mathfrak{so}(3)}{SO(3)} \cong T\mathbb{R}^2 \times \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$$\left. \begin{aligned} \dot{q}^1 - \omega_{q^2} &= -\Omega q^2, \\ \dot{q}^2 + \omega_{q^1} &= \Omega q^1, \\ \omega_{q^3} &= c \end{aligned} \right\} (**)$$

with $c \in \mathbb{R}$ and $(\omega_{q^1}, \omega_{q^2}, \omega_{q^3})$ the components of the angular velocity of the sphere

$$\ddot{q}^1 + \frac{k^2 \Omega}{1 + k^2} \dot{q}^2 = 0, \quad \ddot{q}^2 - \frac{k^2 \Omega}{1 + k^2} \dot{q}^1 = 0$$

Optimization problem

Full control over the motion of the center of the ball

$$l(q^1, q^2, \dot{q}^1, \dot{q}^2; \omega_{q^1}, \omega_{q^2}, \omega_{q^3}) = \frac{1}{2}((\dot{q}^1)^2 + (\dot{q}^2)^2)$$

PLATE-BALL PROBLEM *Given points $q_0, q_1 \in Q$, find an optimal control curve $(q^1(t), q^2(t))$ on the reduced space that steer the system from q_0 to q_1 , minimizes $\int_0^1 \frac{1}{2} ((\dot{q}^1)^2 + (\dot{q}^2)^2) dt$, subject to the constraints defined by Equations (**)*

Koon, Marsden (97)

$$E \equiv \text{The Atiyah algebroid } \tau_E : T\mathbb{R}^2 \times \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$$\pi : C = \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow Q = \mathbb{R}^2, \quad ((q^1, q^2), (u^1, u^2)) \rightarrow (q^1, q^2)$$

$$\sigma : C \equiv \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow E = T\mathbb{R}^2 \times \mathbb{R}^3$$

$$((q^1, q^2), (u^1, u^2)) \rightarrow (q^1, q^2; u^1, u^2, -u^2 + \Omega q^1, u^1 + \Omega q^2, c)$$

The index function $\equiv l$

$$v^1 = \dot{q}^1, \quad v^2 = \dot{q}^2, \quad v^3 = \omega_{q^1}, \quad v^4 = \omega_{q^2}, \quad v^5 = \omega_{q^3}$$

$$\dot{p}_3 = cp_4 - (v^1 + \Omega q^2) p_5$$

$$\dot{p}_4 = -cp_3 - (v^2 - \Omega q^1) p_5$$

$$\dot{p}_5 = (v^1 + \Omega q^2) p_3 + (v^2 - \Omega q^1) p_4$$

$$\frac{d}{dt} (v^1 - p_4) = -\Omega p_3$$

$$\frac{d}{dt} (v^2 + p_3) = -\Omega p_4$$

SOME CONCLUSIONS

- The theory of Lie algebroids plays an important role in the geometric formulation of variational nonholonomic (vakonomic) Lagrangian systems with symmetries
- The theory of Lie algebroids gives a natural interpretation for *the use of quasi-coordinates in vakonomic mechanics*

To take local coordinates which are adapted to the vakonomic problem (they simplify the resolution of the dynamical equations)



To choose an appropriated local basis of sections of the Lie algebroid

(Reduction of subriemannian problems)