A momentum-energy integrator for nonholonomic mechanical systems with symmetry

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Discrete Euler–Lagrange equations

- A *discrete Lagrangian* is a map $L_d \colon Q \times Q \to \mathbb{R}$, which should be thought of as an approximation of a continuous Lagrangian $L \colon TQ \to \mathbb{R}$.
- For *unconstrained* systems, a discrete variational principle yields the well-known discrete Euler–Lagrange (DEL) equations

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = 0.$$

- One can find q_{k+1} from q_{k-1} and q_k if and only if the matrix $(D_{12} L_d)$ is regular, i.e., if the discrete Lagrangian is regular.

A geometric nonholonomic integrator

- From now on, we will work with nonholonomic mechanical systems with constraints linear in the velocities, which are given by a distribution $\mathcal{D}$ in TQ.
- We restrict ourselves to the case where the Lagrangian is of mechanical type

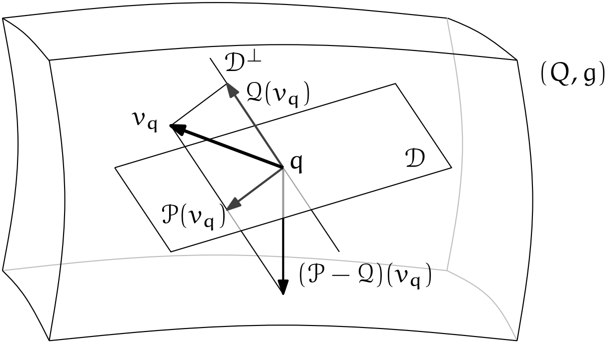
  $$L(v_q) = \frac{1}{2} g(v_q, v_q) - V(q), \quad v_q \in T_q Q,$$

  where g is a Riemannian metric on Q.

A geometric nonholonomic integrator

By using the metric g of the kinetic energy, we can obtain two complementary projectors

$$\mathcal{P}\colon TQ \to \mathcal{D}$$
$$\mathcal{Q}\colon TQ \to \mathcal{D}^{\perp}.$$



A geometric nonholonomic integrator

- The proposed discrete nonholonomic equations are

$$\mathcal{P}^*_{|q_k}(D_1L_d(q_k, q_{k+1})) + \mathcal{P}^*_{|q_k}(D_2L_d(q_{k-1}, q_k)) = 0$$
$$\mathcal{Q}^*_{|q_k}(D_1L_d(q_k, q_{k+1})) - \mathcal{Q}^*_{|q_k}(D_2L_d(q_{k-1}, q_k)) = 0$$

  - The first equation is the projection of DEL to $\mathcal{D}$.
  - The second equation can be interpreted as an elastic impact of the system against $\mathcal{D}$. It is a discretization of the constraints.

- Alternatively,

$$D_1L_d(q_k, q_{k+1}) + (\mathcal{P}^* - \mathcal{Q}^*)_{|q_k}(D_2L_d(q_{k-1}, q_k)) = 0$$

- This defines a unique discrete evolution operator if and only if the matrix $(D_{12}L_d)$ is regular, i.e., if the discrete Lagrangian is regular (same condition as for unconstrained systems).

A geometric nonholonomic integrator

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Example: nonholonomic particle

- Continuous Lagrangian $L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$
- Nonholonomic constraint: $\dot{z} - y\dot{x} = 0$.
- Discrete Lagrangian:

$$L_d = \frac{h}{2}\left[\left(\frac{x_1 - x_0}{h}\right)^2 + \left(\frac{y_1 - y_0}{h}\right)^2 + \left(\frac{z_1 - z_0}{h}\right)^2\right].$$

- Discrete nonholonomic equations:

$$\begin{aligned} \left(\frac{x_2 - 2x_1 + x_0}{h^2}\right) + y_1\left(\frac{z_2 - 2z_1 + z_0}{h^2}\right) &= 0 \\ \frac{y_2 - 2y_1 + y_0}{h^2} &= 0 \\ \frac{z_2 - z_0}{2h} - y_1\frac{x_2 - x_0}{2h} &= 0. \end{aligned}$$

- DLA algorithm (Cortés and Martínez, 2001) replaces the third equation by

$$\frac{z_2 - z_1}{h} - \left(\frac{y_2 + y_1}{2}\right)\frac{x_2 - x_1}{h} = 0.$$

A geometric nonholonomic integrator

Define the discrete Legendre transformations associated to L_d as mappings $Q \times Q \to T^*Q$ given by

$$\begin{aligned} \mathbb{F}^- L_d(q_0, q_1) &= (q_0, -D_1 L_d(q_0, q_1)) \\ \mathbb{F}^+ L_d(q_0, q_1) &= (q_1, D_2 L_d(q_0, q_1)), \end{aligned}$$

and define the pre- and post-momenta

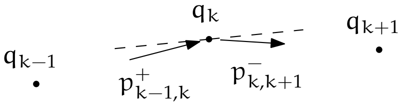
$$\begin{aligned} p^+_{k-1,k} &= \mathbb{F}^+ L_d(q_{k-1}, q_k) = (q_k, D_2 L_d(q_{k-1}, q_k)) \\ p^-_{k,k+1} &= \mathbb{F}^- L_d(q_k, q_{k+1}) = (q_k, -D_1 L_d(q_k, q_{k+1})). \end{aligned}$$

We may rewrite the discrete nonholonomic equations as

$$p^-_{k,k+1} = (\mathcal{P}^* - \mathcal{Q}^*)\big|_{q_k}(p^+_{k-1,k})$$

A geometric nonholonomic integrator

In our method, the momenta are related by a reflection with respect to the image of $\mathcal{P}^*\colon T^*Q \to (\mathcal{D}^\perp)^o \subset T^*Q$.



The second equation of the method,

$$\mathcal{Q}^*_{|q_k}(D_1L_d(q_k, q_{k+1})) - \mathcal{Q}^*_{|q_k}(D_2L_d(q_{k-1}, q_k)) = 0,$$

implies

$$\mathcal{Q}^*_{|q_k}\left(\frac{p^-_{k,k+1} + p^+_{k-1,k}}{2}\right) = 0.$$

In this sense the proposed numerical method preserves the nonholonomic constraints.

Left-invariant systems

- Consider a discrete nonholonomic Lagrangian system on a Lie group G, with a discrete Lagrangian $L_d \colon G \times G \to \mathbb{R}$ that is left-invariant.
- Define the increment $W_k = g_k^{-1} g_{k+1}$. The pre- and post-momenta turn out to be related by

  $$p^+_{k,k+1} = R^*_{W_k^{-1}} p^-_{k,k+1}$$

  where R^* is the mapping on T^*G induced by right multiplication.
- Therefore, the discrete nonholonomic equations become

  $$p^-_{k,k+1} = (\mathcal{P} - \mathcal{Q})^* \left(R^*_{W_{k-1}^{-1}} p^-_{k-1,k} \right).$$

Left-invariant systems

- Suppose that both L_d and $\mathcal{D}$ are left-invariant.
- Define the discrete body momentum as

$$p_k = L^*_{g_k} p^-_{k,k+1} \in \mathfrak{g}^*.$$

- Our method becomes

$$p_k = (\mathcal{P} - \mathcal{Q})^* \left(\mathrm{Ad}^*_{W_{k-1}} p_{k-1} \right).$$

- Compare to the discrete Euler–Poincaré–Suslov equations (Fedorov and Zenkov, 2005):

$$p_k = \mathrm{Ad}^*_{W_{k-1}} p_{k-1} + \sum_j \lambda^j a^j$$

Preserving energy on Lie groups

Theorem

If

- *the configuration manifold is a Lie group,*
- *the continuous Lagrangian is bi-invariant,*
- *the discrete Lagrangian is left-invariant, and*
- $\mathcal{D}$ *is arbitrary,*

then

- *the proposed method is energy-preserving.*

Proof.

$$L(v_g) = \frac{1}{2}\left\langle \mathbb{I} g^{-1} v_g, g^{-1} v_g \right\rangle,$$

where $\mathbb{I}\colon \mathfrak{g} \to \mathfrak{g}^*$ is a non-singular, equivariant inertia tensor: $\mathrm{Ad}^*_{g^{-1}} \circ \mathbb{I} = \mathbb{I} \circ \mathrm{Ad}_g$ for all $g \in G$.

Preserving energy on Lie groups

Proof (cont'd). There is a bi-invariant inner product and a corresponding norm $\|\cdot\|_{\mathbb{I}}$ on each fiber of T^*G:

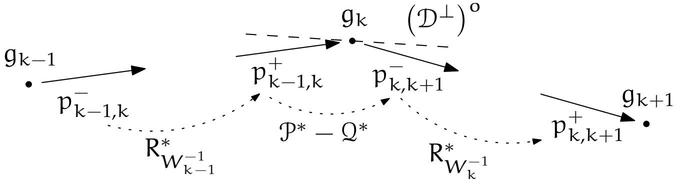
$$\|p_g\|_{\mathbb{I}}^2 = \left\langle p_g, p_g^\sharp \right\rangle = \left\langle p_g, L_g \mathbb{I}^{-1} L_g^* p_g \right\rangle = \left\langle p_g, R_g \mathbb{I}^{-1} R_g^* p_g \right\rangle .$$

$\mathcal{P}^*$ and $\mathcal{Q}^*$ are orthogonal complementary projectors with respect to this inner product, and thus for $p \in T^*G$,

$$\begin{aligned} \|(\mathcal{P} - \mathcal{Q})^* p\|_{\mathbb{I}}^2 &= \langle \mathcal{P}^* p, \mathcal{P}^* p \rangle_{\mathbb{I}} + \langle \mathcal{Q}^* p, \mathcal{Q}^* p \rangle_{\mathbb{I}} = \|(\mathcal{P} + \mathcal{Q})^* p\|_{\mathbb{I}}^2 \\ &= \|p\|_{\mathbb{I}}^2 . \end{aligned}$$

Preserving energy on Lie groups

Proof (cont'd).



The energy is $H(g,p) = \frac{1}{2}\|p\|^2_{\mathbb{I}}$. Since $R^*_{W^{-1}_{k-1}}$ and $(\mathcal{P}-\mathcal{Q})^*$ are norm-preserving, the evolution equation

$$p^-_{k,k+1} = (\mathcal{P}-\mathcal{Q})^*\left(R^*_{W^{-1}_{k-1}} p^-_{k-1,k}\right)$$

preserves $\|\cdot\|_{\mathbb{I}}$ and $H(g_k, p^-_{k,k+1}) = H(g_{k-1}, p^-_{k-1,k})$. □

Discrete nonholonomic momentum map

- Consider a Lie group G acting on Q.
- Define, for each $q \in Q$, the vector subspace $\mathfrak{g}^q$ consisting of those elements of $\mathfrak{g}$ whose infinitesimal generators at q satisfy the nonholonomic constraints, i.e.,

  $$\mathfrak{g}^q = \left\{ \xi \in \mathfrak{g} \,|\, \xi_Q(q) \in \mathcal{D}_q \right\} .$$

  We collect these in a bundle over Q denoted by $\mathfrak{g}^{\mathcal{D}}$.
- Define the discrete nonholonomic momentum map $J_d^{nh} \colon Q \times Q \to (\mathfrak{g}^{\mathcal{D}})^*$:

  $$\begin{aligned} J_d^{nh}(q_{k-1}, q_k) \colon \mathfrak{g}^{q_k} &\to \mathbb{R} \\ \xi &\mapsto \left\langle D_2 L_d(q_{k-1}, q_k), \xi_Q(q_k) \right\rangle . \end{aligned}$$

- For any smooth section $\widetilde{\xi}$ of $\mathfrak{g}^{\mathcal{D}}$ we have a function $(J_d^{nh})_{\widetilde{\xi}} \colon Q \times Q \to \mathbb{R}$, defined as

  $$(J_d^{nh})_{\widetilde{\xi}}(q_{k-1}, q_k) = J_d^{nh}(q_{k-1}, q_k) \left(\widetilde{\xi}(q_k) \right) .$$

Discrete nonholonomic momentum map

Theorem

Assume that L_d *is* G*-invariant, and let* $\widetilde{\xi}$ *be a smooth section of* $\mathfrak{g}^{\mathcal{D}}$*. Then* $(J_d^{nh})_{\widetilde{\xi}}$ *evolves according to the equation*

$$(J_d^{nh})_{\widetilde{\xi}}(q_k, q_{k+1}) - (J_d^{nh})_{\widetilde{\xi}}(q_{k-1}, q_k) = \left\langle D_2 L_d(q_k, q_{k+1}), \left(\widetilde{\xi}(q_{k+1}) - \widetilde{\xi}(q_k)\right)_Q (q_{k+1}) \right\rangle$$

A horizontal symmetry is an element $\xi \in \mathfrak{g}$ such that $\xi_Q(q) \in \mathcal{D}_q$ for all $q \in Q$.

Corollary

If L_d *is* G*-invariant and* ξ *is a horizontal symmetry, then the proposed nonholonomic integrator preserves* $(J_d^{nh})_\xi$.

Nonholonomic version of the Störmer–Verlet method

- On $Q = \mathbb{R}^n$, consider the Lagrangian

  $$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M\dot{q} - V(q)$$

  and the constraints

  $$\mu(q)\dot{q} = 0$$

  where $\mu(q)$ is a $m \times n$ matrix with rank $\mu = m$
- Symmetric discretization:

  $$\begin{aligned} L_d(q_k, q_{k+1}) &= \frac{h}{2}L\left(q_k, \frac{q_{k+1} - q_k}{h}\right) + \frac{h}{2}L\left(q_{k+1}, \frac{q_{k+1} - q_k}{h}\right) \\ &= \frac{1}{2h}\left(q_{k+1} - q_k\right)^T M\left(q_{k+1} - q_k\right) \\ &\quad - \frac{h}{2}\left(V(q_k) + V(q_{k+1})\right) \end{aligned}$$

Nonholonomic version of the Störmer–Verlet method

Equations:

$$q_{k+1} - 2q_k + q_{k-1} = -h^2 M^{-1} \left(V_q(q_k) + \mu^T(q_k)\widetilde{\lambda}_k \right)$$

$$0 = \mu(q_k) \left(\frac{q_{k+1} - q_{k-1}}{2h} \right)$$

Extension to the SHAKE method proposed by Ryckaert, Ciccotti and Berendsen (1977), to the case of nonholonomic constraints, which is a generalization of the classical Störmer–Verlet method in presence of holonomic constraints. These equations also appear in McLachlan and Perlmutter (2006).

Nonholonomic version of the Störmer–Verlet method

- Momenta:

$$\widetilde{p}_k = \frac{1}{2}\left(p^+_{k-1,k} + p^-_{k,k+1}\right) = M(q_{k+1} - q_{k-1})/2h$$

$$p_{k+1/2} = M(q_{k+1} - q_k)/h$$

- Rewriting the equations in the previous slide we get

$$\begin{aligned} p_{k+1/2} &= \widetilde{p}_k - \frac{h}{2}\left(V_q(q_k) + \mu^T(q_k)\widetilde{\lambda}_k\right), \\ q_{k+1} &= q_k + hM^{-1}p_{k+1/2}, \\ 0 &= \mu(q_k)M^{-1}\widetilde{p}_k, \\ \widetilde{p}_{k+1} &= M(q_{k+2} - q_k)/2h. \end{aligned}$$

- Replace last equation by an additional step of the algorithm:

$$\begin{aligned} \widetilde{p}_{k+1} &= p_{k+1/2} - \frac{h}{2}\left(V_q(q_{k+1}) + \mu^T(q_{k+1})\widetilde{\lambda}_{k+1}\right), \\ 0 &= \mu(q_{k+1})M^{-1}\widetilde{p}_{k+1}. \end{aligned}$$

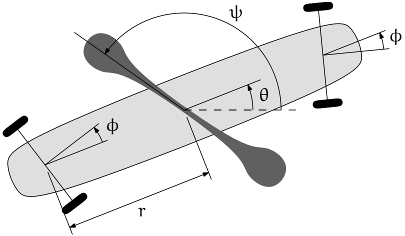
Nonholonomic version of the Störmer–Verlet method

- We obtain

$$\begin{aligned} p_{k+1/2} &= \widetilde{p}_k - \frac{h}{2}\left(V_q(q_k) + \mu^T(q_k)\widetilde{\lambda}_k\right) \\ q_{k+1} &= q_k + hM^{-1}p_{k+1/2} \\ 0 &= \mu(q_k)M^{-1}\widetilde{p}_k \\ \widetilde{p}_{k+1} &= p_{k+1/2} - \frac{h}{2}\left(V_q(q_{k+1}) + \mu^T(q_{k+1})\widetilde{\lambda}_{k+1}\right) \\ 0 &= \mu(q_{k+1})M^{-1}\widetilde{p}_{k+1} \end{aligned}$$

- Extension of the RATTLE algorithm for holonomic systems to the case of nonholonomic systems.
- Natural constraint on initial conditions: $\mu(q_0)M^{-1}\widetilde{p}_0 = 0$.

Example: the snakeboard



- $Q = SE(2) \times \mathbb{T}^2$ with coordinates $q = (x, y, \theta, \psi, \phi)$.

- Continuous Lagrangian:

$$L(q,\dot{q}) = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}(J+2J_1)\dot{\theta}^2 + \frac{1}{2}J_0\dot{\psi}^2 + J_1\dot{\phi}^2$$

$m =$ total mass, $J =$ m. i. of the board, $J_0 =$ m. i. of the rotor, $J_1 =$ m. i. of each wheel axle.

- Constraints:

$$\begin{aligned} \dot{x}\sin(\theta+\phi) - \dot{y}\cos(\theta+\phi) + r\dot{\theta}\cos(\phi) &= 0 \\ \dot{x}\sin(\theta-\phi) - \dot{y}\cos(\theta-\phi) - r\dot{\theta}\cos(\phi) &= 0. \end{aligned}$$

Example: the snakeboard

- Define the discrete Lagrangian

$$L_d(q_k, q_{k+1}) = \frac{1}{2h^2}\Big(m(\Delta x_k^2 + \Delta y_k^2) + (J + 2J_1)\Delta\theta_k^2 \\ + J_0\Delta\psi_k^2 + 2J_1\Delta\phi_k^2\Big)$$

  Here $\Delta z_k = z_{k+1} - z_k$.
- $SE(2) \times \mathbb{T}^2$ is a Lie group.
- The Lagrangian is *not* bi-invariant (only left-invariant), so preservation of energy is not guaranteed.

But...

- Change the group structure from $SE(2)$ to $\mathbb{R}^2 \times S^1$.
- Then L and L_d are bi-invariant.
- The numerical method itself does not depend on which group structure one takes.
- There is preservation of energy.

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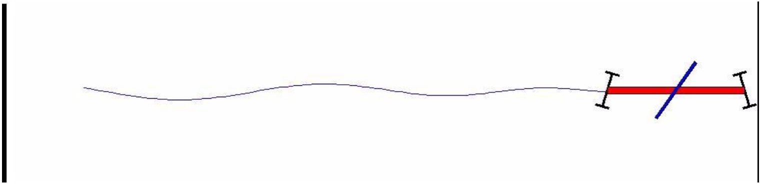
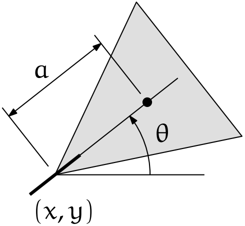
Example: the snakeboard

Add controls on the right-hand side of the equations:

- equal but opposite torques on the wheel axles;
- torque on the rotor.



Example: Chaplygin sleigh



- $Q = SE(2)$ with coordinates $q = (x, y, \theta)$.
- $a =$ distance from contact point to center of mass.

- Continuous Lagrangian:

$$L = \frac{1}{2} m \left(\dot{x}^2 - 2a\dot{\theta}\dot{x}\sin(\theta) + \dot{y}^2 + 2a\dot{\theta}\dot{y}\cos(\theta) + a^2\dot{\theta}^2 \right) + \frac{1}{2} I \dot{\theta}^2$$

- Constraints:

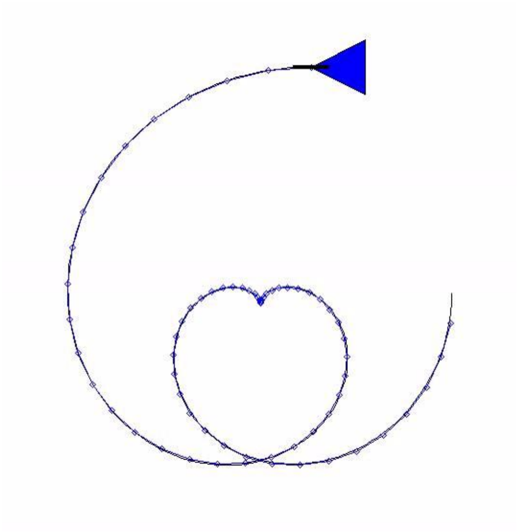
$$\dot{x}\sin(\theta) - \dot{y}\cos(\theta) = 0$$

- Discrete Lagrangian:

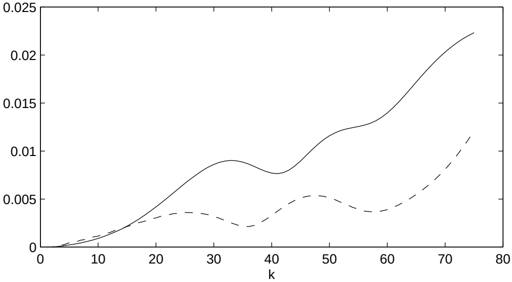
$$\dot{z} \rightsquigarrow \frac{z_{k+1} - z_k}{h}, \qquad z \rightsquigarrow \frac{z_{k+1} + z_k}{2}$$

Example: Chaplygin sleigh

Comparison with correct trajectory



Example: Chaplygin sleigh



Error in $\mathbb{R}^3$ of the trajectories computed with our method (dashed line) and DLA (solid).

Further reading

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