New developments in Geometric Integration of Nonholonomic Systems

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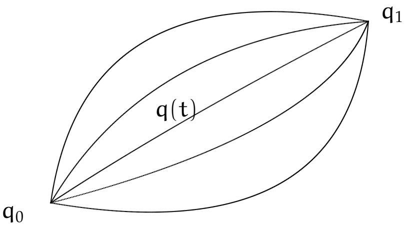
> *"...The problem for the more general class of non-holonomic constraints is still open, as is the question of the correct analogue of symplectic integration for non-holonomically constrained Lagrangian systems...".*

McLachlan-Scovel:A survey of open problems in symplectic integration

Hamilton's principle

Obtain numerical integrators from a discrete version of variational Hamilton's principle

Define the set $C^2(q_0, q_1; [t_0, t_1])$ as the C^2-curves $\sigma : [t_0, t_1] \to Q$ such $q(t_0) = q_0$ and $q(t_1) = q_1$.



$$\begin{array}{rccc} F: & C^2(q_0, q_1; [t_0, t_1]) & \longrightarrow & \mathbb{R} \\ & q(\cdot) & \longmapsto & \int_{t_1}^{t_0} L(t, q(t), \dot{q}(t))\, dt \end{array}$$

L lagrangian function $L : \mathbb{R} \times TQ \to \mathbb{R}$.

Hamilton's principle $\Longrightarrow \delta \int_{t_0}^{t_1} L(t, q(t), \dot{q}(t))\, dt = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0,\ 1 \leqslant i \leqslant n = \dim Q$$

$$L_d(q_0, q_1, t_0, t_1) \approx \int_{t_0}^{t_1} L(t, q(t), \dot{q}(t))\, dt$$

$$L_d(q_0, q_1, h) \approx \int_{0}^{h} L(t, q(t), \dot{q}(t))\, dt$$

For h fixed, $L_d : Q \times Q \longrightarrow \mathbb{R}$

DISCRETE MECHANICS

Q n-dimensional differentiable manifold $L_d : Q \times Q \longrightarrow \mathbb{R}$ discrete lagrangian.

Action: $S : Q^{N+1} \longrightarrow \mathbb{R}$

$$S = \sum_{k=0}^{N-1} L_d(q_k, q_{k+1})$$

Discrete Euler-Lagrange equations (DEL))

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = 0$$

$$\Downarrow$$

$$\begin{array}{rccc} F_{L_d} : & Q \times Q & \longrightarrow & Q \times Q \\ & (q_0, q_1) & \longmapsto & (q_1, q_2) \end{array}$$

Discrete flow

Momentum preservation

L_d is G-invariant

$$J_d : Q \times Q \longrightarrow \mathfrak{g}^*$$

$$\langle J_d(x,y), \xi \rangle = \langle D_2 L_d(x,y), \xi_Q(y) \rangle$$

$$\langle D_2 L_d(q_{k-1}, q_k), \xi_Q(q_k) \rangle = \langle D_2 L_d(q_k, q_{k+1}), \xi_Q(q_{k+1}) \rangle$$

Symplecticity

$$\begin{array}{rccc} \mathbb{F}^{-}L_d : & Q \times Q & \longrightarrow & T^*Q \\ & (x,y) & \longmapsto & (x, -D_1L_d(x,y)) \end{array}$$

$$\begin{array}{rccc} \mathbb{F}^{+}L_d : & Q \times Q & \longrightarrow & T^*Q \\ & (x,y) & \longmapsto & (y, D_2L_d(x,y)) \end{array}$$

$$\omega_{L_d} = (\mathbb{F}^{\pm}L_d)^*\omega_Q$$

The discrete flow preserves the symplectic form $F^*_{L_d}(\omega_{L_d}) = \omega_{L_d}$

VARIATIONAL INTEGRATORS

If the discrete Lagrangian is an approximation to a continuous Lagrangian then the discrete system is an numerical integrator for the continous system.

- **Symplectic**
- **Momentum preserving**

E. Hairer, C. Lubic and G. Wanner: *Geometric Numerical Integration, Structure-Preserving Algorithms for Ordinary Differential Equations*, Springer Series in Computational Mathematics **31**, Springer-Verlag Berlin Heidelberg, 2002.

J. E. Marsden and M. West: *Discrete mechanics and variational integrators*, *Acta Numerica* 2001, 357-514.

Reduction of mechanical systems: Euler-Poincaré reduction.

Let G be a Lie group and let G act on itself by left translation and hence, by tangent lift, on its tangent bundle TG. Let $L : TG \to \mathbb{R}$ be a G-invariant Lagrangian.
Then, L is completely determined by its restriction to the tangent space at the identity e. Identifying T_eG with the Lie algebra $\mathfrak{g}$ of G, we define $l : \mathfrak{g} \to \mathbb{R}$.
The velocity of the system is given by $\dot{g}(t)$, thought of as a tangent vector to G at $g(t)$. The body velocity is defined by $\xi(t) = g(t)^{-1}\dot{g}(t)$, the left translation of $\dot{g}(t)$ to the identity.
Theorem (Euler-Poincaré reduction)
A curve $g(t)$ in G satisfies the Euler-Lagrange equations for L iff $\xi(t)$ satisfies the Euler-Poincaré equations for l:

$$\frac{d}{dt}\frac{\partial l}{\partial \xi} = ad^*_{\xi}\frac{\partial l}{\partial \xi}$$

Reduction of mechanical systems: Lagrange-Poincaré reduction.

Assume Q is Riemannian (the metric often being the kinetic energy metric) and that G acts on Q freely by isometries, so $Q \to Q/G$ is a principal bundle. If we declare the horizontal spaces to be metric orthogonal to the group orbits, this uniquely defines a connection called the mechanical connection. The space Q/G is called shape space.

$TQ/G \cong T(Q/G) \oplus \tilde{\mathfrak{g}}$

If we have an invariant Lagrangian on TQ it induces a Lagrangian $\mathfrak{l}$ on $(TQ)/G$ and hence on $T(Q/G) \oplus \tilde{\mathfrak{g}}$.

Reduced Euler-Lagrange equations

$$L : TQ \longrightarrow \mathbb{R}$$

$$L : \mathfrak{g} \longrightarrow \mathbb{R}$$

$$L : (TQ)/G \longrightarrow \mathbb{R}$$

IS IT POSSIBLE TO CONSTRUCT VARIATIONAL INTEGRATORS FOR REDUCED SYSTEMS?

$$TQ \rightsquigarrow Q \times Q$$

Mechanics on Lie groupoids

The pair or banal groupoid

$$Q \times Q$$

$$\begin{array}{lcl}
\alpha : Q \times Q \longrightarrow Q & ; & (q, q') \longrightarrow q, \\
\beta : Q \times Q \longrightarrow Q & ; & (q, q') \longrightarrow q', \\
\epsilon : Q \longrightarrow Q \times Q & ; & q \longrightarrow (q, q), \\
m : (Q \times Q)_2 \longrightarrow Q \times Q & ; & (q, q'), (q', q'') \longrightarrow (q, q''), \\
i : Q \times Q \longrightarrow Q \times Q & ; & (q, q') \longrightarrow (q', q).
\end{array}$$

The product manifold $Q \times Q \rightrightarrows Q$ is a Lie groupoid over Q. If q is a point of Q, it follows that

$$V_{\epsilon(q)}\alpha = \{0_q\} \times T_qQ \subseteq T_qQ \times T_qQ \cong T_{(q,q)}(Q \times Q).$$

Thus, the linear maps

$$\Psi_q : T_qQ \to V_{\epsilon(q)}\alpha, \quad v_q \to (0_q, v_q),$$

induce an isomorphism (over the identity of Q).

Lie groupoids

A **groupoid** over a set M is a set G together with the following structure maps:

- A pair of maps $\alpha : G \to M$, the **source**, and $\beta : G \to M$, the **target**. These maps define the set of composable pairs

$$G_2 = \{(g, h) \in G \times G / \beta(g) = \alpha(h)\}.$$

- A **multiplication** $m : G_2 \to G$, to be denoted simply by $m(g, h) = gh$, such that
  - $\alpha(gh) = \alpha(g)$ and $\beta(gh) = \beta(h)$.
  - $g(hk) = (gh)k$.
- An **identity section** $\epsilon : M \to G$ such that
  - $\epsilon(\alpha(g))g = g$ and $g\epsilon(\beta(g)) = g$.
- An **inversion map** $i : G \to G$, to be denoted simply by $i(g) = g^{-1}$, such that
  - $g^{-1}g = \epsilon(\beta(g))$ and $gg^{-1} = \epsilon(\alpha(g))$.

The groupoid $G \rightrightarrows M$ is said to be a **Lie groupoid** if G and M are manifolds and all the structure maps are differentiable with α and β differentiable submersions.

If $x \in M$, $\alpha^{-1}(x)$ (resp., $\beta^{-1}(x)$) will be said the *α-fiber* (resp., the *β-fiber*) of x.

If $g \in G$ then the **left-translation** by $g \in G$ and the **right-translation** by g are the diffeomorphisms

$$\begin{array}{ll} l_g : \alpha^{-1}(\beta(g)) \longrightarrow \alpha^{-1}(\alpha(g)), \quad ; & h \longrightarrow l_g(h) = gh, \\ r_g : \beta^{-1}(\alpha(g)) \longrightarrow \beta^{-1}(\beta(g)), \quad ; & h \longrightarrow r_g(h) = hg. \end{array}$$

A vector field $\tilde{X}$ on G is said to be **left-invariant** (resp., **right-invariant**) if it is tangent to the fibers of α (resp., β) and $\tilde{X}(gh) = (T_h l_g)(\tilde{X}_h)$ (resp., $\tilde{X}(gh) = (T_g r_h)(\tilde{X}(g))$, for $(g, h) \in G_2$.

Lie algebroid associated with G

We consider the vector bundle $\tau : AG \to M$, whose fiber at a point $x \in M$ is

$$A_x G = V_{\epsilon(x)}\alpha = \mathrm{Ker}(T_{\epsilon(x)}\alpha)$$

It is easy to prove that there exists a bijection between the space $\Gamma(\tau)$ an the set of left-invariant (resp., right-invariant) vector fields on G. If X is a section of $\tau : AG \to M$, the corresponding left-invariant (resp., right-invariant) vector field on G will be denoted $\overleftarrow{X}$ (resp., $\overrightarrow{X}$).

$$\overleftarrow{X}(g) = (T_{\epsilon(\beta(g))} l_g)(X(\beta(g))),$$

$$\overrightarrow{X}(g) = (T_{\epsilon(\alpha(g))} r_g)\{X(\alpha(g)) - (T_{\alpha(g)}\epsilon)((T_{\epsilon(\alpha(g))}\beta)(X(\alpha(g))))\},$$

Using the above facts, we may introduce a Lie algebroid structure on AG:

$$\overleftarrow{[\![X, Y]\!]} = [\overleftarrow{X}, \overleftarrow{Y}], \quad \rho(X)(x) = (T_{\epsilon(x)}\beta)(X(x)),$$

for $X, Y \in \Gamma(\tau)$ and $x \in M$.
Note that

$$\overrightarrow{[\![X, Y]\!]} = -[\overrightarrow{X}, \overrightarrow{Y}], \quad [\overrightarrow{X}, \overleftarrow{Y}] = 0,$$

Given a section X of $\tau : AG \longrightarrow M$, then the corresponding left and right-invariant representatives are related by the inversion map:

$$Ti \circ \overleftarrow{X} = -\overrightarrow{X} \circ i \qquad \text{and} \qquad Ti \circ \overrightarrow{X} = -\overleftarrow{X} \circ i.$$

Examples of Lie groupoids

1. **Lie groups** G.

$$\alpha(g) = \mathfrak{e}, \ \ \beta(g) = \mathfrak{e}, \ \ \epsilon(\mathfrak{e}) = \mathfrak{e}, \ \ i(g) = g^{-1}, \ \ m(g, h) = gh$$

$$G \rightsquigarrow \mathfrak{g}$$

2. **The banal groupoid** $Q \times Q$.

$$\alpha(x, y) = x, \ \ \beta(x, y) = y, \ \ \epsilon(x) = (x, x), \ \ i(x, y) = (y, x)$$
$$m((x, y), (y, z)) = (x, z)$$

$$Q \times Q \rightsquigarrow TQ$$

3 **Atiyah or gauge groupoids**. Let $p : Q \to M$ be a principal G-bundle. Then, the free action, $\Phi : G \times Q \to Q$, $(g, q) \to \Phi(g, q) = gq$, of G on Q induces, in a natural way, a free action $\Phi \times \Phi : G \times (Q \times Q) \to Q \times Q$ of G on $Q \times Q$ given by $(\Phi \times \Phi)(g, (q, q')) = (gq, gq')$, for $g \in G$ and $(q, q') \in Q \times Q$. Moreover, one may consider the quotient manifold $(Q \times Q)/G$ and it admits a Lie groupoid structure over M with structural maps given by

$$\begin{array}{lcl}
\tilde{\alpha} : (Q \times Q)/G \longrightarrow M & ; & [(q, q')] \longrightarrow p(q), \\
\tilde{\beta} : (Q \times Q)/G \longrightarrow M & ; & [(q, q')] \longrightarrow p(q'), \\
\tilde{\epsilon} : M \longrightarrow (Q \times Q)/G & ; & x \longrightarrow [(q, q)], \quad \text{if } p(q) = x, \\
\tilde{m} : ((Q \times Q)/G)_2 \longrightarrow (Q \times Q)/G & ; & ([(q, q')], [(gq', q'')]) \longrightarrow [(gq, q'')], \\
\tilde{\imath} : (Q \times Q)/G \longrightarrow (Q \times Q)/G & ; & [(q, q')] \longrightarrow [(q', q)].
\end{array}$$

This Lie groupoid is called *the Atiyah (gauge) groupoid associated with the principal G-bundle* $p : Q \to M$.

$$(Q \times Q)/G \rightsquigarrow (TQ)/G$$

DISCRETE LAGRANGIAN MECHANICS

A **discrete lagrangian** is a function $L : G \longrightarrow \mathbb{R}$.

$$\mathcal{C}_g^N = \{(g_1, \dots, g_N) \in G^N \ / \ (g_k, g_{k+1}) \in G_2 \text{ for each } k = 1, \dots, N-1 \text{ and } g_1 \dots g_n = g\}$$

$$c(t) = (g_1 h_1(t), h_1^{-1}(t) g_2 h_2(t), \dots, h_{N-2}^{-1}(t) g_{N-1} h_{N-1}(t), h_{N-1}^{-1}(t) g_N)$$

where $h_k(t) \in \alpha^{-1}(\beta(g_k))$ and $h_k(0) = \epsilon(\beta(g_k))$ for $k = 1, \dots, N-1$.

$$T_{(g_1, g_2, \dots, g_N)} \mathcal{C}_g^N \equiv \left\{ (X_1, X_2, \dots, X_{N-1})) | X_k \in A_{x_k} G \text{ and } x_k = \beta(g_k), 1 \leqslant k \leqslant N-1 \right\}$$

Discrete action sum

$$\begin{array}{rccc} \mathcal{S}L \ : & \mathcal{C}_g^N & \longrightarrow & \mathbb{R} \\ & (g_1, \dots, g_N) & \longmapsto & \displaystyle\sum^{N} L(g_k) \end{array}$$

Discrete Euler-Lagrange equations

[Discrete Hamilton's principle]

An admissible sequence $(g_1, \ldots, g_N) \in \mathcal{C}_g^N$ is a solution of the Lagrangian system determined by $L : G \longrightarrow \mathbb{R}$ if and only if $(g_1, \ldots, g_N)$ is a critical point of $\mathcal{S}L$.

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \mathcal{S}L(c(t)) &= \frac{d}{dt}\Big|_{t=0} \Big\{ L(g_1 h_1(t)) + L(h_1^{-1}(t) g_2 h_2(t)) \\
&\quad + \ldots + L(h_{N-2}^{-1}(t) g_{N-1} h_{N-1}(t)) + L(h_{N-1}^{-1}(t) g_N) \Big\} \\
&= \sum_{k=1}^{N-1} \Big(d(L \circ l_{g_k})(x_k)(X_k) + d(L \circ r_{g_{k+1}} \circ i)(x_k)(X_k) \Big)
\end{aligned}$$

Critical condition

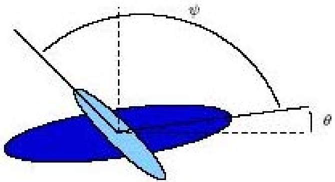
$$0 = \overleftarrow{X}_k\big|_{g_k}(L) - \overrightarrow{X}_k\big|_{g_{k+1}}(L)$$

J.C. Marrero, D. Martín de Diego, E. Martínez: *Discrete Lagrangian and Hamiltonian Mechanics on Lie groupoids*, Nonlinearity 2006.

- We introduce two Poincaré-Cartan 1-sections Θ_L^+ and Θ_L^-, and an unique Poincaré-Cartan 2-section, Ω_L.
- We study the discrete Lagrangian evolution operator $\xi : G \longrightarrow G$ and its preservation properties.
- Reduction theory is established in terms of morphisms of Lie groupoids.
- The associated Hamiltonian formalism is developed using the discrete Legendre transformations $\mathbb{F}^+L : G \to A^*G$ and $\mathbb{F}^-L : G \to A^*G$.
- A complete characterization of the regularity of a Lagrangian on a Lie groupoid is given in terms of the symplecticity of Ω_L.
- We prove a Noether's theorem for discrete Mechanics on Lie groupoids.

Example: Elroy's beanie

This system is probably the more simple example of a dynamical system with a non-Abelian Lie group symmetries. It consists in two planar rigid bodies attached at their centers of mass, moving freely in the plane.



Configuration space: The configuration space is $Q = SE(2) \times S^1$ with coordinates (x, y, θ, ψ), where the three first coordinates describe the position and orientation of the center of mass of the first body and the last one the relative orientation between both bodies.

Lagrangian function. We consider the Lagrangian

$$L(x, y, \theta, \psi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\psi}) = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)+\frac{1}{2}I_1\dot{\theta}^2+\frac{1}{2}I_2(\dot{\theta}+\dot{\psi})^2-V(\psi)$$

where m denotes the mass of the system and I_1 and I_2 are the inertias of the first body and the second body, respectively; additionally, we also consider a potential function of the form $V(\psi)$.

Symmetry. The symmetry group we consider is $G = SE(2)$. The Lagrangian L is $SE(2)$-invariant.

Continuous system

$$l^R : TQ/SE(2) \longrightarrow \mathbb{R}$$

$$l^R(\psi, \dot{\psi}, \Omega) = \frac{1}{2} m(\Omega_1^2 + \Omega_2^2) + \frac{1}{2}(I_1 + I_2)\Omega_3^2 + \frac{1}{2}\frac{I_1 I_2}{I_1 + I_2}\dot{\psi}^2 - V(\psi)$$

Lagrange-Poincaré equations

$$\left\{ \begin{array}{rcl} \dot{\Omega}_1 & = & \Omega_2\Omega_3 - \frac{I_2}{I_1+I_2}\dot{\psi}\Omega_2 \\ \dot{\Omega}_2 & = & -\Omega_1\Omega_3 + \frac{I_2}{I_1+I_2}\dot{\psi}\Omega_1 \\ \dot{\Omega}_3 & = & 0 \\ \frac{I_1 I_2}{I_1+I_2}\ddot{\psi} & = & -\dfrac{\partial V}{\partial \psi} \end{array} \right\}$$

Discrete system

We introduce the discrete Lagrangian l_d^R on $(Q \times Q)/SE(2)$ given by

$$
\begin{aligned}
l_d^R(\psi_k, \psi_{k+1}, \Omega_{(1)k}, \Omega_{(2)k}, \Omega_{(3)k}) &= \frac{1}{2h^2} m \left[\Omega^2_{(1)k} + \Omega^2_{(2)k}\right] \\
&+ \frac{I_1 + I_2}{h^2} \left[1 - \cos(\Omega_{(3)k})\right] + \frac{1}{2} \frac{I_1 I_2}{I_1 + I_2} \left(\frac{\Delta\psi_k}{h}\right)^2 - V\left(\frac{\psi_k + \psi_{k+1}}{2}\right)
\end{aligned}
$$

Discrete Lagrange-Poincaré equations

$$
\left\{
\begin{aligned}
&\Omega_{(1)k+1} = \Omega_{(1)k} \cos(\Omega_{(3)k} + \tfrac{I_2}{I_1+I_2} \Delta\Psi_k) - \Omega_{(2)k} \sin(\Omega_{(3)k} + \tfrac{I_2}{I_1+I_2} \Delta\Psi_k) \\
&\Omega_{(2)k+1} = \Omega_{(1)k} \sin(\Omega_{(3)k} + \tfrac{I_2}{I_1+I_2} \Delta\Psi_k) + \Omega_{(2)k} \cos(\Omega_{(3)k} + \tfrac{I_2}{I_1+I_2} \Delta\Psi_k) \\
&\Omega_{(3)k+1} = \Omega_{(3)k} \\
&\frac{I_1 I_2}{I_1 + I_2} \frac{\psi_{k+2} - 2\psi_{k+1} + \psi_k}{h^2} = -\frac{1}{2} \left(\frac{\partial V}{\partial \psi}\left(\frac{\psi_{k+2} + \psi_{k+1}}{2}\right) + \frac{\partial V}{\partial \psi}\left(\frac{\psi_{k+1} + \psi_k}{2}\right) \right)
\end{aligned}
\right.
$$

Discrete generalized Holder's principle

$G \rightrightarrows M$ a Lie groupoid; $\dim G = m+n, \quad \dim M = m$

$\alpha, \beta : G \to M, \ \ \epsilon : M \to G; i : G \to G, \quad m : G_2 \to G$

$\tau : AG \to M \equiv$ the Lie algebroid of G

Generalized discrete nonholonomic (or constrained) Lagrangian system

- $L_d : G \to \mathbb{R}$ a *regular discrete Lagrangian*
- *The constraint distribution* $\mathcal{D}_c$
  $\tau_{\mathcal{D}_c} : \mathcal{D}_c \to M$ a vector subbundle of AG, $\qquad \mathrm{rank}\mathcal{D}_c = r$
- *The discrete constraint embedded submanifold* $\mathcal{M}_c$
  $i_{\mathcal{M}_c} : \mathcal{M}_c \to G$ is a embedded submanifold of G

Assumption

$$\dim \mathcal{M}_c = \dim \mathcal{D}_c = m + r, \quad r \leqslant n$$

$(L_d, \mathcal{M}_c, \mathcal{D}_c) \equiv$ *a discrete nonholonomic Lagrangian system on* G

$g \in G$ fixed

$$\mathcal{C}_g^N = \{(g_1, \ldots, g_N) \in G^N / (g_k, g_{k+1}) \in G_2, \text{ for } k = 1, .., N-1 \text{ and } g_1 \ldots g_N = g\}$$

$$T_{(g_1, g_2, .., g_N)} \mathcal{C}_g^N \equiv \{(v_1, v_2, \ldots, v_{N-1}) \,|\, v_k \in (AG)_{x_k} \text{ and } x_k = \beta(g_k), 1 \leqslant k \leqslant N-1\}$$

Discrete action sum

$$\mathcal{S}L_d : \mathcal{C}_g^N \longrightarrow \mathbb{R} \quad (g_1, \ldots, g_N) \longmapsto \sum_{k=1}^{N} L_d(g_k)$$

$$(\mathcal{V}_c)_{(g_1, ..., g_N)} = \{(v_1, \ldots, v_{N-1}) \in T_{(g_1, ..., g_N)} \mathcal{C}_g^N / \forall k \in \{1, \ldots, N-1\}, \, v_k \in \mathcal{D}_c\}$$

Discrete Hölder's principle

Discrete Hölder's principle

$g \in \Gamma, \quad (g_1, \dots, g_N) \in \mathcal{C}_g^N$

$(g_1, \dots, g_N)$ is a solution of the discrete nonholonomic Lagrangian system: $(L_d, \mathcal{M}_c, \mathcal{D}_c)$ if

$$\Updownarrow$$

- $g_k \in \mathcal{M}_c, \quad \forall k \in \{1, \dots, N\}$
- $\delta \mathcal{S} L_d|_{(\mathcal{V}_c)_{g_1 \dots, g_N}} = 0$

$$(g_1, \ldots, g_N) \in \mathcal{C}_g^N$$

$$\Downarrow$$

- $g_k \in \mathcal{M}_c, \quad \forall k \in \{1, \ldots, N\}$
- $\displaystyle\sum_{k=1}^{N-1} (d^o(L_d \circ l_{g_k}) + d^o(L_d \circ r_{g_{k+1}} \circ i))(\epsilon(\beta(g_k))_{|(\mathcal{D}_c)(\beta(g_k))} = 0$

$\beta(g_k) = \alpha(g_{k+1}) = x_k$

Discrete Nonholonomic equations

$N = 2, \quad (g, h) \in \Gamma_2, \quad \beta(g) = \alpha(h) = x$

$$(g, h) \text{ is a solution}$$

$$\Updownarrow$$

$(g, h) \in \mathcal{M}_c \times \mathcal{M}_c, \qquad d^o(L_d \circ l_g + L_d \circ r_h \circ i)(\epsilon(x))_{|(\mathcal{D}_c)_x} = 0$
Discrete nonholonomic Euler-Lagrange equations for the system $(L_d, \mathcal{M}_c, \mathcal{D}_c)$

D. Iglesias, J.C. Marrero, D. Martin de Diego and E. Martinez: Discrete Nonholonomic Lagrangian Systems on Lie Groupoids. To appear in Journal of Nonlinear Science (2008).

Alternative versions of the discrete nonholonomic E-L equations

$$\{X^\alpha\} \text{ a local basis of } \Gamma(\mathcal{D}_c^0)$$

$$(g,h) \in G_2, \quad \beta(g) = \alpha(h) = x \in M$$

$$(g,h) \text{ is a solution of the discrete nonholonomic problem}$$

$$\Updownarrow$$

$$(g,h) \in \mathcal{M}_c \times \mathcal{M}_c$$

$$d^o\left[L_d \circ l_g + L_d \circ r_h \circ i\right](\epsilon(x))(v) = \lambda_\alpha X^\alpha(x)(v),$$

$$\lambda_\alpha \equiv \text{ the Lagrange multipliers}$$

Alternative versions of the discrete nonholonomic E-L equations

$\{X_a\}$ a local basis of sections in $\Gamma(\mathcal{D}_c)$

$$(g, h) \in \mathcal{M}_c \times \mathcal{M}_c, \quad \beta(g) = \alpha(h) = x \in M$$

(g, h) is a solution of the discrete nonholonomic problem

$$\boxed{0 = \overleftarrow{X}_a\Big|_{g_k}(L) - \overrightarrow{X}_a\Big|_{g_{k+1}}(L), , \forall a}$$

The standard case: $G = Q \times Q$

$$\Gamma = Q \times Q$$

$$(q_0, q_1) \in \mathcal{M}_c$$

$$((q_0, q_1), (q_1, q_2)) \text{ is a solution}$$

$$\Updownarrow$$

$$(q_1, q_2) \in \mathcal{M}_c$$

$$D_2 L_d(q_0, q_1) + D_1 L_d(q_1, q_2) = \lambda_\alpha A^\alpha(q_1)$$

Cortés, Martínez (2001)
McLachlan, Perlmuther(2006)

Discrete nonholonomic equations on Lie groups, G

$$g_1 \in \mathcal{M}_c$$

$(g_1, g_2) \in G \times G$ is a solution of the discrete nonholonomic Euler-Lagrange equations for $(L_d, \mathcal{M}_c, \mathcal{D}_c)$

$$\Updownarrow$$

$$\begin{aligned} g_1^{-1} dL_d(g_1) - dL_d(g_2) g_2^{-1} &= \sum_{j=1}^{n-r} \lambda^j \mu_j, \\ g_1 &\in \mathcal{M}_c \end{aligned}$$

λ^j the Lagrange multipliers
$\{\mu_j\}$ a basis of $\mathcal{D}_c^0$

Notation: $g, h \in G$, $\alpha_h \in T_h^* G$

$$g\alpha_h \;=\; (T^*_{gh} l_{g^{-1}})(\alpha_h) \in T^*_{gh} G, \quad \alpha_h g \;=\; (T^*_{hg} r_{g^{-1}})(\alpha_h) \in T^*_{hg} G.$$

Federov, Zenkov (2005)
McLachLan, Perlmutter (2006)

An example of a discrete nonoholonomic Lagrangian systems on an Atiyah Lie groupoid.

"A (homogeneous) sphere of radius $r > 0$*, mass* m *and inertia about any axis I rolls without sliding on a horizontal table which rotates with constant angular velocity* Ω *about a vertical axis through one of its points"*

- Configuration space

$$Q = \mathbb{R}^2 \times SO(3), \qquad (x, y; R) \in Q$$

- The Lagrangian function $L : TQ \to \mathbb{R}$

$$L(x, y; R, \dot{x}, \dot{y}; \dot{R}) = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{4} I \operatorname{tr}(\dot{R}R^T (\dot{R}R^T)^T)$$

- The constrained submanifold $\mathcal{M}$

$$\mathcal{M} = \{(x, y; R, \dot{x}, \dot{y}; \dot{R}) / \qquad \begin{array}{l} \dot{x} + \frac{r}{2}\operatorname{tr}(\dot{R}R^T E_L) = -\Omega y \\ \dot{y} - \frac{r}{2}\operatorname{tr}(\dot{R}R^T E_1) = \Omega x \} \end{array}$$

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \; E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \; E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

the standard basis of $\mathfrak{so}(3)$

The constrained system is $SO(3)$-invariant

$$\Downarrow$$

$(L', \mathcal{M}')$ a constrained Lagrangian system on the corresponding Atiyah algebroid

$$E' \cong TQ/SO(3) \longrightarrow \mathbb{R}^2$$

- The vector bundle $E' = TQ/SO(3) \cong T\mathbb{R}^2 \times \mathfrak{so}(3) \longrightarrow \mathbb{R}^2$
- The anchor map $\rho' : E' \cong T\mathbb{R}^2 \times \mathfrak{so}(3) \longrightarrow T\mathbb{R}^2$ the projection over the first factor
- The Lie bracket on $Sec(E')$

$$[\![s_3', s_4']\!]' = s_5', \quad [\![s_4', s_5']\!]' = s_3', \quad [\![s_5', s_3']\!]' = s_4',$$

$\{s_i'\}_{i=1,\ldots,5}$ the canonical basis of $Sec(E')$

- The reduced Lagrangian function:

$$L'(x, y, \dot{x}, \dot{y}; w) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{4}I\mathrm{tr}(w^2)$$

- The reduced constraint submanifold

$$\mathcal{M}' = \{(x, y, \dot{x}, \dot{y}; w) / \dot{x} + \frac{r}{2}\mathrm{tr}(wE_2) = -\Omega y, \dot{y} - \frac{r}{2}\mathrm{tr}(wE_1) = \Omega x\}$$

Objective: To discretize the nonholonomic Lagrangian system $(L', \mathcal{M}')$ on the Atiyah algebroid $E' \cong T\mathbb{R}^2 \times \mathfrak{so}(3) \to \mathbb{R}^2$
The discrete Atiyah groupoid $G' \cong \mathbb{R}^2 \times \mathbb{R}^2 \times SO(3) \rightrightarrows \mathbb{R}^2$
The discrete Lagrangian function Using an approximation of the (local) inverse of the exponential map

$$\exp : \mathfrak{so}(3) \to SO(3)$$

$$\Downarrow$$

$$L'_d(x_0, y_0, x_1, y_1; W_1) = \frac{1}{2} m[(\frac{x_1 - x_0}{h})^2 + (\frac{y_1 - y_0}{h})^2] + \frac{I}{(2h)^2} tr(W_1)$$

The *discrete constraint submanifold* $\mathcal{M}'_c$

$$\begin{array}{rcl} \dfrac{x_1 - x_0}{h} + \dfrac{r}{2h}\mathrm{tr}(W_1 E_2) & = & -\Omega \dfrac{y_1 + y_0}{2}, \\ \dfrac{y_1 - y_0}{h} - \dfrac{r}{2h}\mathrm{tr}(W_1 E_1) & = & \Omega \dfrac{x_1 + x_0}{2}, \end{array}$$

The discrete constraint distribution

$$\mathcal{D}'_c =< \{s'_5, rs'_1 + s'_4, rs'_2 - s'_3\} >$$

The discrete constrained Euler-Lagrange equations for $(L'_d, \mathcal{M}'_c, \mathcal{D}'_c)$

$$\begin{aligned}
\frac{x_2 - 2x_1 + x_0}{h^2} + \frac{I\Omega}{I + mr^2}\frac{y_2 - y_0}{2h} &= 0 \\
\frac{y_2 - 2y_1 + y_0}{h^2} - \frac{I\Omega}{I + mr^2}\frac{x_2 - x_0}{2h} &= 0 \\
\mathrm{tr}\left((W_1 - W_2)E_3\right) &= 0 \\
\frac{x_2 - x_1}{h} + \frac{r}{2h}\mathrm{tr}(W_2 E_2) + \Omega\frac{y_2 + y_1}{2} &= 0, \\
\frac{y_2 - y_1}{h} - \frac{r}{2h}\mathrm{tr}(W_2 E_1) - \Omega\frac{x_2 + x_1}{2} &= 0
\end{aligned}$$

$(x_0, y_0, x_1, y_1; W_1)$ are known

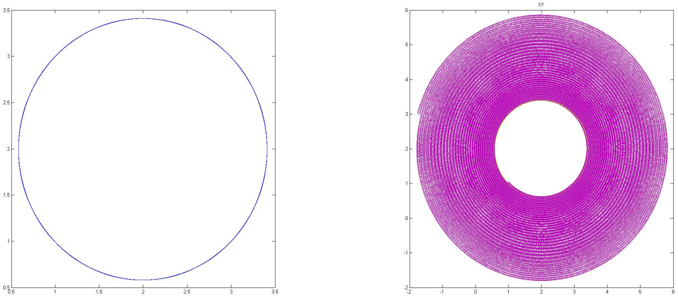


Figure: Orbits for the discrete nonholonomic equations of motion (left) and a standard numerical method (right) (initial conditions $x_0 = 0.99$, $y_0 = 1$, $x_1 = 1$, $y_1 = 0.99$ and $h = 0.01$ after 20000 steps).

Other numerical integrator for nonholonomic systems

S. Ferraro, D. Iglesias, D. Martín de Diego: Momentum and energy preserving integrators for nonholonomic dynamics, *arXiv:0709.1463v2*

Admits an extension to Lie groupoids integrators!
Let $\mathcal{G} : AG \times_M AG \to \mathbb{R}$ be a bundle metric on a Lie algebroid $(AG, [\cdot\,,\cdot], \rho)$.
The class of systems that were considered is that of *mechanical systems with nonholonomic constraints* determined by:

- The Lagrangian function L:

  $$L(a) = \frac{1}{2}\mathcal{G}(a, a) - V(\tau(a)), \quad a \in AG,$$

  with V a function on M.
- The nonholonomic constraints determined by a subbundle $\mathcal{D}$ of AG,

Consider the orthogonal decomposition $AG = \mathcal{D} \oplus \mathcal{D}^{\perp}$, and the associated orthogonal projectors

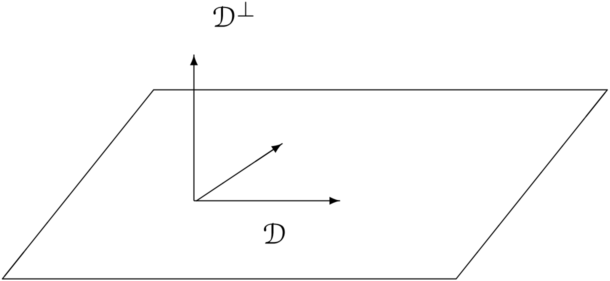
$$P: \; AG \longrightarrow \mathcal{D}$$

$$Q: \; AG \longrightarrow \mathcal{D}^{\perp}$$

Given local coordinates (x^i) in the base manifold M and a local basis of sections of AG (moving basis), $\{X_A\}$, adapted to the nonholonomic problem $(L, \mathcal{D})$, in the sense that

(i) $\{X_A\}$ is an orthonormal basis with respect to $\mathcal{G}$ (that is $\mathcal{G}(X_A, X_B) = \delta_{AB}$)

(ii) $\{X_A\} = \{X_a, X_\alpha\}$ where $\mathcal{D} = \text{span}\{X_a\}$, $\mathcal{D}^{\perp} = \text{span}\{X_\alpha\}$.



Denoting by $(x^i, y^A) = (x^i, y^a, y^\alpha)$ the induced coordinates on AG, the constraint equations determining $\mathcal{D}$ just read $\boxed{y^A = 0.}$ Therefore we choose (x^i, y^a) as a set of coordinates on $\mathcal{D}$.

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, hook, "i_{\mathcal{D}}"] \arrow[dr, "\tau_{\mathcal{D}}"'] & & AG \arrow[dl, "\tau"] \\
& M &
\end{tikzcd}$$

In this coordinates we have the inclusion

$$\begin{array}{rccc} i_{\mathcal{D}}: & \mathcal{D} & \longrightarrow & AG \\ & (x^i, y^a) & \longmapsto & (x^i, y^a, 0) \end{array}$$

and the dual map

$$\begin{array}{rccc} i_D^*: & A^*G & \longrightarrow & D^* \\ & (x^i, y_a, y_\alpha) & \longmapsto & (x^i, y_a) \end{array}$$

where (x^i, y_A) are the induced coordinates on A^*G by the dual

Moreover from the orthogonal decomposition we have that

$$\begin{array}{rccc} P: & AG & \longrightarrow & \mathcal{D} \\ & (x^i, y^a, y^\alpha) & \longmapsto & (x^i, y^a) \end{array}$$

and its dual map

$$\begin{array}{rccc} P^*: & \mathcal{D}^* & \longrightarrow & A^*G \\ & (x^i, y_a) & \longmapsto & (x^i, y_a, 0) \end{array}$$

Linear almost Poisson structures and Hamilton-Jacobi theory. Applications to nonholonomic Mechanics. Manuel de León, Juan Carlos Marrero, D. Martín de Diego, *arXiv:0801.4358v1 [math-ph].*

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Nonholonomic Integrator

$L_d : G \longrightarrow \mathbb{R}$ discretization of L.

$$\boxed{\begin{aligned} 0 &= \overleftarrow{X}_a\big|_{g_k}(L_d) - \overrightarrow{X}_a\big|_{g_{k+1}}(L_d), \forall a \\ 0 &= \overleftarrow{X}_\alpha\big|_{g_k}(L_d) + \overrightarrow{X}_\alpha\big|_{g_{k+1}}(L_d), \forall \alpha \end{aligned}}$$

and more work in progress.........

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and more work in progress.........

Thank you very much for your attention