CARLESON MEASURES FOR THE BLOCH SPACE

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Abstract. In this paper we study the positive Borel measures μ on the unit disc $\mathbb{D}$ in $\mathbb{C}$ for which the Bloch space $\mathcal{B}$ is continuously included in $L^p(d\mu)$, $0 < p < \infty$. We call such measures p-Bloch-Carleson measures. We give two conditions on a measure μ in terms of certain logarithmic integrals one of which is a necessary condition and the other a sufficient condition for μ being a p-Bloch-Carleson measure. We also give a complete characterization of the p-Bloch-Carleson measures within certain special classes of measures. It is also shown that, for $p > 1$, the p-Bloch-Carleson measures are exactly those for which the Toeplitz operator T_μ, defined by $T_\mu(f)(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^2}\, d\mu(w)$ $(f \in L^1(d\mu), z \in \mathbb{D})$, maps continuously $L^{p'}(d\mu)$ into the Bergman space A^1, $\frac{1}{p} + \frac{1}{p'} = 1$. Furthermore, we prove that if $p > 1$, $\alpha > -1$ and ω is a weight which satisfies the Bekollé-Bonami $\mathcal{B}_{p,\alpha}$-condition, then the measure $\mu_{\alpha,p}$ defined by $d\mu_{\alpha,p}(z) = (1 - |z|^2)^\alpha \omega(z) dA(z)$ is a p-Bloch-Carleson-measure.

We also consider the Banach space $H^\infty_{\log}$ of those functions f which are analytic in $\mathbb{D}$ and satisfy $|f(z)| = \mathrm{O}\left(\log \frac{1}{1-|z|}\right)$, as $|z| \to 1$. The Bloch space is contained in $H^\infty_{\log}$. We describe the p-Carleson measures for $H^\infty_{\log}$ and study weighted composition operators and a class of integration operators acting in this space. We determine which of these operators map $H^\infty_{\log}$ continuously to the weighted Bergman space A^p_α $(p > 0, \alpha > -1)$ and show that they are automatically compact.

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1. Introduction and main results

Let $\mathbb{D}$ denote the open unit disc of the complex plane $\mathbb{C}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{H}ol(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$. If $0 < r < 1$ and $f \in \mathcal{H}ol(\mathbb{D})$, we set

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p\,dt\right)^{1/p}, \quad 0 < p < \infty,$$
$$M_\infty(r,f) = \sup_{|z|=r} |f(z)|.$$

For $0 < p < \infty$, $\alpha > -1$ the Bergman space A^p_α is the set of all $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\int_{\mathbb{D}} |f(z)|^p\,dA_\alpha(z) < \infty,$$

where $dA_\alpha(z) = \pi^{-1}(1+\alpha)(1-|z|^2)^\alpha dx\,dy = \pi^{-1}(1+\alpha)r(1-r^2)^\alpha\,dr\,d\theta$. We shall write simply $dA(z)$ for $dA_0(z)$ and A^p for the classical Bergman space A^p_0. We mention [17] and [25] as general references for the theory of Bergman spaces.

The Bloch space $\mathcal{B}$ consists of those $f \in \mathcal{H}ol(\mathbb{D})$ for which

$$\rho_{\mathcal{B}}(f) \stackrel{\text{def}}{=} \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

The Bloch space is a Banach space with the norm $\|\cdot\|_{\mathcal{B}}$ defined by

$$\|f\|_{\mathcal{B}} = |f(0)| + \rho_{\mathcal{B}}(f), \quad f \in \mathcal{B}.$$

A general reference for the Bloch space is [3].

If $I \subset \partial\mathbb{D}$ is an interval, $|I|$ will denote the length of I. The *Carleson square* $S(I)$ is defined as

$$S(I) = \{re^{it} : e^{it} \in I, \quad 1 - \frac{|I|}{2\pi} \le r < 1\}.$$

If $s > 0$ and μ is a positive Borel measure in $\mathbb{D}$, we shall say that μ is an s-Carleson measure if there exists a positive constant C such that

$$\mu\left(S(I)\right) \le C|I|^s, \quad \text{for any interval } I \subset \mathbb{T}.$$

An 1-Carleson measure will be simply called a (classical) Carleson measure.

If X is a subspace of $\mathcal{H}ol(\mathbb{D})$, $0 < p < \infty$, and μ is a positive Borel measure in $\mathbb{D}$, μ is said to be a p-Carleson measure for the space X if $X \subset L^p(d\mu)$. For a large class of spaces $X \subset \mathcal{H}ol(\mathbb{D})$ a characterization of the p-Carleson measures for the space X is known and such a characterization is useful in the study of the boundedness and the compactness of operators acting on X. Denote by H^p $(0 < p \le \infty)$

the classical Hardy spaces of analytic functions in $\mathbb{D}$ (see [15]). Carleson [11] characterized the p-Carleson measures for the Hardy space H^p. Namely, he proved that $H^p \subset L^p(d\mu)$, $0 < p < \infty$, if and only if μ is a Carleson measure. The q-Carleson measures for the space H^p were characterized by Duren [14] in the case $0 < p < q < \infty$, and by Luecking [28] in the case $0 < q < p < \infty$ (see also the recent paper [9]). Luecking [26, 29], characterized the q-Carleson measures for the space A^p_α, $0 < p, q < \infty$. A good number of results about p-Carleson measures for Besov spaces and spaces of Dirichlet type of analytic functions have been obtained by different authors (see, e. g., [4], [5], [20], [21], [22], [33], [39], [42], [43], and [44]).

Our main purpose in this paper is studying the Carleson measures for the Bloch space. A p-Carleson measure for the Bloch space will be simply called a *p-Bloch-Carleson measure*. Using the closed graph theorem, we see that μ is a p-Bloch-Carleson measure if and only if there exists a positive constant such that

$$\int_{\mathbb{D}} |f(z)|^p \, d\mu(z) \le C\|f\|^p_{\mathcal{B}} \tag{1.1}$$

for all $f \in \mathcal{B}$. Let us remark that since all constant functions are in $\mathcal{B}$, a p-Bloch-Carleson measure is necessarily a finite measure.

Let $H^\infty_{\log}$ denote the space of those functions $f \in \mathcal{H}ol(\mathbb{D})$ such that $M_\infty(r,f) = \mathrm{O}\left(\log \frac{1}{1-r}\right)$, as $r \to 1$. The space $H^\infty_{\log}$ is a Banach space with the norm $||\cdot||_{H^\infty_{\log}}$ defined by $||f||_{H^\infty_{\log}} = \sup_{z\in\mathbb{D}} \frac{|f(z)|}{\log \frac{e}{1-|z|}} < \infty$.

It is clear that if $0 < p < \infty$ and μ is a positive Borel measure in $\mathbb{D}$ such that

$$\int_{\mathbb{D}} \left(\log \frac{1}{1-|z|}\right)^p d\mu(z) < \infty, \tag{1.2}$$

then μ is a p-Carleson measure for the space $H^\infty_{\log}$.

Since it is also clear that $\mathcal{B} \subset H^\infty_{\log}$, it follows that (1.2) implies that μ is a p-Bloch-Carleson measure.

On the other hand, using an argument of Arazy, Fisher and Peetre [4] we shall prove that if μ is a p-Bloch-Carleson measure then

$$\int_{\mathbb{D}} \left(\log \frac{1}{1-|z|}\right)^{p/2} d\mu(z) < \infty. \tag{1.3}$$

That is, we have:

Theorem 1. *Suppose that $0 < p < \infty$ and μ is a positive Borel measure in $\mathbb{D}$. Then:*

(i) *If μ satisfies* (1.2) *then μ is a p-Bloch-Carleson measure.*

(ii) *If μ is a p-Bloch-Carleson measure then μ satisfies* (1.3).

It is natural to ask whether either (1.2) or (1.3) is equivalent to saying that μ is a p-Bloch-Carleson measure.

We recall that Ramey and Ullrich [35] proved that there exist two Bloch functions f_1 and f_2 such that

$$(1-|z|^2)\left(|f_1'(z)|+|f_2'(z)|\right)\geq 1, \quad \text{for all } z\in\mathbb{D}.$$

If we were able to find two functions $f_1, f_2 \in \mathcal{B}$ such that

$$|f_1(z)|+|f_2(z)|\geq \log\frac{1}{1-|z|}, \quad z\in\mathbb{D},$$

then condition (1.2) would characterize p-Bloch-Carleson measures. However, such a pair of functions does not exist: Its existence would be in contradiction with the estimate

$$M_p(r,f)=\mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{1/2}\right), \text{ as } r\to 1, \tag{1.4}$$

valid for all $f\in\mathcal{B}$ and $0<p<\infty$, which was proved by Clunie and MacGregor [12] and Makarov [30].

In any case, one may ask the following questions.

Question 1. Do there exist two functions $f_1, f_2 \in \mathcal{B}$ such that

$$|f_1(z)|+|f_2(z)|\geq \left(\log\frac{1}{1-|z|}\right)^{1/2}, \quad z\in\mathbb{D}?$$

Question 2. Do there exist two functions $f_1, f_2 \in H^\infty_{\log}$ such that

$$|f_1(z)|+|f_2(z)|\geq \log\frac{1}{1-|z|}, \quad z\in\mathbb{D}?$$

We do not know the answer to Question 1. However, we will show that the answer to Question 2 is affirmative.

Theorem 2. *There exist two functions $f_1, f_2 \in H^\infty_{\log}$ such that*

$$|f_1(z)|+|f_2(z)|\geq \log\frac{1}{1-|z|}, \quad z\in\mathbb{D}.$$

As a consequence we shall deduce that the p-Carleson measures for the space $H^\infty_{\log}$ are characterized by condition (1.2):

Theorem 3. *Suppose that $0<p<\infty$ and μ is a positive Borel measure in $\mathbb{D}$. Then μ is a p-Carleson measure for the space $H^\infty_{\log}$ if and only if it satisfies* (1.2).

Theorem 2 and Theorem 3 will be proved in Section 8 where we shall prove other results about the space $H^\infty_{\log}$ which may be of independent interest. In particular, we shall characterize the membership in $H^\infty_{\log}$ of a class of functions given by power series with large gaps. These results will be applied in Section 9 to study the boundedness and compactness of a number of operators acting on $H^\infty_{\log}$. In particular, we shall give a complete characterization of the weighted composition operators which map $H^\infty_{\log}$ continuously into A^p_α and will show that they are all compact.

Back to the Bloch space, the estimate (1.4) might suggest that the condition (1.3) characterizes the p-Bloch-Carleson measures. However, we shall prove that this is not the case at all. In fact, we shall see that neither the converse of (i) nor the converse of (ii) is true in general. On the other hand, we shall find large classes of Borel measures μ in $\mathbb{D}$ for which (1.3) or (1.2) is equivalent to saying that μ is a p-Bloch-Carleson measure. All of this will be found in Section 5. More precisely, the following results related to condition (1.3) will be proved there.

Theorem 4. *Suppose that $0 < p < \infty$, $0 < r < 1$ and μ is a positive Borel measure in $\mathbb{D}$ for which there exists a positive constant C such that*

$$\mu\left(\Delta(\lambda_1, r)\right) \le C\mu\left(\Delta(\lambda_2, r)\right) \tag{1.5}$$

for every pair of points $\lambda_1, \lambda_2 \in \mathbb{D}$ with $|\lambda_1| = |\lambda_2|$. Then μ is a p-Bloch-Carleson measure if and only if μ satisfies (1.3).

Here, $\Delta(a, r)$ denotes the pseudohyperbolic disc of center $a \in \mathbb{D}$ and radius r. We remark that if μ is radial, that is,

$$d\mu(re^{i\theta}) = d\nu(r), \;\; \theta \in [0, 2\pi),$$

for a certain positive Borel measure ν in $(0, 1)$, then μ satisfies condition (1.5) of Theorem 4. Hence, we have:

Corollary 1. *Suppose that $0 < p < \infty$ and μ is a positive Borel measure in $\mathbb{D}$ which is radial, that is,*

$$d\mu(re^{i\theta}) = d\nu(r), \;\; \theta \in [0, 2\pi),$$

for a certain positive Borel measure ν in $(0, 1)$. Then μ is a p-Bloch-Carleson measure if and only if μ satisfies (1.3).

We shall also find a large class of discrete measures μ in $\mathbb{D}$ for which μ is p-Bloch-Carleson measure if and only if it satisfies condition (1.2). One of our results of this kind is the following.

Theorem 5. *Suppose that $0 < p < \infty$ and $\{a_k\}_{k=1}^{\infty}$ is a sequence of positive numbers. Let*

$$\mu = \sum_{k=1}^{\infty} a_k \delta_{1-e^{-k}},$$

where, as usual, δ_{z_k} denotes the point mass at z_k. Then the following assertions are equivalent:
(i) $\sum_{k=1}^{\infty} a_k k^p < \infty$.
(ii) *μ satisfies* (1.2).
(iii) *μ is a p-Bloch-Carleson measure.*

Corollary 2. *Suppose that $0 < p < \infty$ and $\{a_k\}_{k=1}^{\infty}$ is a sequence of positive numbers such that*

$$\sum_{k=1}^{\infty} a_k k^{p/2} < \infty, \quad \textit{and} \quad \sum_{k=1}^{\infty} a_k k^p = \infty.$$

Set $\mu = \sum_{k=1}^{\infty} a_k \delta_{1-e^{-k}}$. Then μ satisfies (1.3) *and it is not a p-Bloch-Carleson measure.*

The discrete measure μ constructed in Theorem 5 has its support contained in a radius. In Theorem 10 we shall construct sequences $\{z_k\}$ in $\mathbb{D}$ having every point $\xi \in \partial\mathbb{D}$ as an accumulation point and with the property that if $\{a_k\}$ is a sequence of positive numbers and $\mu = \sum a_k \delta_{z_k}$, then

$$\mu \text{ is a } p\text{-Bloch-Carleson measure} \Leftrightarrow \mu \text{ satisfies (1.2).}$$

Section 6 will be devoted to study the relationship between α-Carleson measures and p-Bloch-Carleson measures. We shall prove the following result.

Theorem 6. (a) *If μ is an α-Carleson measure in $\mathbb{D}$ for some $\alpha > 1$, then μ is a p-Bloch-Carleson measure for all $p > 0$.*

(b) *For every $p \in (0, \infty)$ there exists a classical Carleson measure μ in $\mathbb{D}$ which is not a p-Bloch-Carleson measure. More precisely, given $p \in (0, \infty)$ there exists a function $g \in \mathcal{H}ol(\mathbb{D})$ such that the measure $\mu_{g,p}$ in $\mathbb{D}$ given by $d\mu_{g,p}(z) = (1 - |z|^2)^{p-1} |g'(z)|^p \, dA(z)$ is not a p-Bloch-Carleson measure but is a classical Carleson measure.*

(c) *For every $p \in (0, \infty)$ there exists a p-Bloch-Carleson measure which is not an α-Carleson measure for any $\alpha > 0$.*

Next we shall restrict ourselves to the case $p > 1$. For this range of values of p we shall use duality arguments to obtain results which are stronger than those in Theorem 1.

Hence, assume that $1 < p < \infty$ and let p' be conjugate exponent of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. Let μ be a positive Borel measure in $\mathbb{D}$. The duality relation $L^p(d\mu) = \left(L^{p'}(d\mu)\right)^*$ immediately yields that

$$\boxed{\mu \text{ satisfies (1.2)} \;\Leftrightarrow\; L^{p'}(d\mu) \subset L^1\left(\log \frac{e}{1-|z|}\, d\mu\right).} \tag{1.6}$$

If μ is a finite positive Borel measure in $\mathbb{D}$, the *Toeplitz operator* T_μ is defined by

$$T_\mu(f)(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^2}\, d\mu(w), \quad z \in \mathbb{D}, \quad f \in L^1(d\mu). \tag{1.7}$$

Zhu [49, Chapter 6] proved that whenever $1 < p < \infty$, T_μ maps A^p continuously into itself if and only if μ is a 2-Carleson measure. Zhao [48] gave a sufficient condition on μ for T_μ to be continuous from A^1 into itself. We shall prove the following result.

Theorem 7. *Suppose that $1 < p < \infty$ and let μ be a positive Borel measure in $\mathbb{D}$. Let p' be conjugate exponent of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. Then the following conditions are equivalent:*

(a) *μ is a p-Bloch-Carleson measure.*

(b) *The operator T_μ maps $L^{p'}(d\mu)$ continuously into the Bergman space A^1.*

A (non-necessarily radial) positive Borel measurable function ω defined in $\mathbb{D}$ will be called a weight. For a weight ω, the weighted Bergman space $A^p_\alpha(\omega)$ consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$||f||^p_{A^p_\alpha(\omega)} \stackrel{\text{def}}{=} \int_{\mathbb{D}} |f(z)|^p \omega(z)\, dA_\alpha(z) < \infty.$$

Definition 1. Suppose that $p > 1$ and $\alpha > -1$. We shall say that the weight ω satisfies the *Bekollé-Bonami $\mathcal{B}_{p,\alpha}$-condition* if there exists a constant $C_\omega > 0$ such that

$$\left(\int_{S(I)} \omega(z)\, dA_\alpha(z)\right)\left(\int_{S(I)} \omega(z)^{\frac{1}{1-p}}\, dA_\alpha(z)\right)^{p-1} \le C_\omega A_\alpha(S(I))^p$$

for every interval $I \subset \partial\mathbb{D}$.

This condition was introduced in [8] and [7] and allows us to identify the dual space of $A^p_\alpha(\omega)$(see [8] or [27]).

The radial weight $\omega(z) = (1-|z|)^\gamma$ satisfies the condition $\mathcal{B}_{p,\alpha}$ if and only if $-1-\alpha < \gamma < (1+\alpha)(p-1)$ (see [5, p. 445] for a similar statement). One can also check that the non-radial weight $\omega(z) =$

$|1-z|^\gamma$ satisfies the condition $\mathcal{B}_{p,\alpha}$ for $\gamma \in (-2-\alpha,(2+\alpha)(p-1)) \setminus \{-1,p-1\}$.

Arcozzi, Rochberg and Sawyer [5] obtained a characterization of the p-Carleson measures for the weighted Besov space

$$B_p(\omega) \stackrel{\text{def}}{=} \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \int_{\mathbb{D}} (1-|z|^2)^{p-2} |f'(z)|^p \omega(z)\, dA(z) < \infty \right\}$$

if ω satisfies a certain regularity condition and the weight $(1-|z|^2)^{p-2}\omega(z)$ satisfies the condition $\mathcal{B}_{p,0}$. We shall prove the following result.

Theorem 8. *Suppose that $p > 1$, $\alpha > -1$ and ω is a weight which satisfies the Bekollé-Bonami $\mathcal{B}_{p,\alpha}$-condition. Then, the measure $\mu_{\alpha,p}$ defined by $d\mu_{\alpha,p}(z) = (1-|z|^2)^\alpha \omega(z) dA(z)$ is a p-Bloch-Carleson measure, or, equivalently,*

$$\mathcal{B} \subset A^p_\alpha(\omega)\,. \tag{1.8}$$

We remark that we do not assume any regularity condition on ω. It is also worth noticing that (1.8) is a generalization of the well-known inclusion $\mathcal{B} \subset A^p_\alpha$.

We close this section noting that, as usual, we shall be using the convention that C will denote a positive constant which may be different at each occurrence.

2. Proof of Theorem 1

We only have to prove (ii). As mentioned above, this will be done by following a reasoning due to Arazy, Fisher and Peetre.

Let $\mathcal{L}$ denote the class of those $f \in \mathcal{H}ol(\mathbb{D})$ which are given by a power series with Hadamard gaps, that is, there exists $\lambda > 1$ such that f is of the form

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \quad (z \in \mathbb{D}), \text{ with } n_{k+1} \geq \lambda n_k \text{ for all } k. \tag{2.1}$$

It is well known (see [3]) that if $f \in \mathcal{L}$ is as in (2.1), then

$$f \in \mathcal{B} \;\Leftrightarrow\; \sup_{k\geq 0} |a_k| < \infty.$$

Thus, if $\lambda \geq 2$ is an integer and $f(z) = \sum_{k=0}^{\infty} z^{\lambda^{k+1}}$, $z \in \mathbb{D}$, then $f \in \mathcal{B}$. This implies that there exists a constant $C > 0$, depending only on λ, such that $\sum_{k=0}^{\infty} r^{\lambda^{k+1}} \leq C \log \frac{1}{1-r}$, for all $r \in [0,1)$. The opposite inequality is also true.

Lemma 1. *Let $1 < \lambda < \infty$ and $0 < r_0 < 1$. Then there exists a positive constant C, depending only on λ and r_0, such that*

$$\sum_{k=0}^{\infty} r^{\lambda^{k+1}} \geq C \log \frac{1}{1-r}$$

for all $r \in [r_0, 1)$.

Proof. Take λ and r_0 as in the statement. Fix $r \in (r_0, 1)$ and define

$$\phi(x) = r^{\lambda^{x+1}}, \ x > 0.$$

Since ϕ is a decreasing function,

$$\sum_{k=0}^{\infty} r^{\lambda^{k+1}} \geq \int_0^{\infty} r^{\lambda^{x+1}}\, dx,$$

and the change of variable

$$x = \log_{\lambda}\left(-\frac{y}{\log r}\right) - 1$$

and the elementary inequality $\log\frac{1}{r} \leq \frac{1-r^2}{r_0}, r_0 < r < 1$, yield

$$\begin{aligned}
\int_0^{\infty} r^{\lambda^{x+1}}\, dx &= \frac{1}{\log\lambda}\int_{-\lambda\log r}^{\infty} e^{-y}\frac{dy}{y} \geq \frac{1}{\log\lambda}\int_{\lambda(1-r^2)r_0^{-1}}^{\infty} e^{-y}\frac{dy}{y} \\
&\geq \frac{1}{\log \lambda e^{\frac{2\lambda}{r_0}}}\int_{\lambda(1-r^2)r_0^{-1}}^{2\frac{\lambda}{r_0}} \frac{dy}{y} \\
&\geq \frac{1}{\log \lambda e^{\frac{2\lambda}{r_0}}}\log\frac{1}{1-r}.
\end{aligned}$$

□

Proof of Theorem 1 (ii). Assume that μ is a p-Bloch-Carleson measure. Then there exists a positive constant C such that (1.1) is satisfied for all $f \in \mathcal{B}$. Choose now $f(z) = \sum_{k=0}^{\infty} z^{\lambda^k}$, where $\lambda \geq 2$ is a fixed natural number. Then $f \in \mathcal{B}$. Replace f by $f_\theta(z) = f(e^{i\theta}z)$, integrate with respect to θ, use Fubini's theorem and Zygmund's result on gap series (see Theorem 8.20 on p. 215 of Volume I of [50]) to obtain

$$\begin{aligned} C\|f\|_{\mathcal{B}}^p &\geq \int_{\mathbb{D}} \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=0}^{\infty} z^{\lambda^k} e^{i\theta\lambda^k} \right|^p d\theta \, d\mu(z) \\ &\asymp \int_{\mathbb{D}} \left(\sum_{k=0}^{\infty} |z|^{2\lambda^k} \right)^{\frac{p}{2}} d\mu(z) \\ &\geq \int_{\mathbb{D}} \left(\sum_{k=0}^{\infty} |z|^{\lambda^{k+1}} \right)^{\frac{p}{2}} d\mu(z), \end{aligned}$$

and the assertion follows by Lemma 1. □

3. The conformally invariant Bloch space

In this section we shall focus in one of the most relevant properties of the Bloch space, that of being a conformally invariant space.

For $a \in \mathbb{D}$, we let the Möbius map $\varphi_a : \mathbb{D} \to \mathbb{D}$ be defined by

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad z \in \mathbb{D}.$$

Then φ_a is an involutive conformal mapping from $\mathbb{D}$ onto itself. Let $\mathrm{Aut}(\mathbb{D})$ denote the group of all conformal mappings from $\mathbb{D}$ onto itself. It is well known that $\mathrm{Aut}(\mathbb{D})$ coincides with the set of all Möbius transformations from $\mathbb{D}$ onto itself:

$$\mathrm{Aut}(\mathbb{D}) = \mathrm{M\ddot{o}b}(\mathbb{D}) = \{\lambda\varphi_a : a \in \mathbb{D}, |\lambda| = 1\}.$$

A space X of analytic functions in $\mathbb{D}$, equipped with a semi-norm ρ, is said to be *conformally invariant* or *Möbius invariant* if whenever $f \in X$, then also $f \circ \varphi \in X$ for any $\varphi \in \mathrm{Aut}(\mathbb{D})$ and, moreover, $\rho(f \circ \varphi) \leq C\rho(f)$ for some positive constant C and all $f \in X$.

The Bloch space is conformally invariant:

$$\rho_{\mathcal{B}}(f \circ \varphi) = \rho_{\mathcal{B}}(f), \quad \text{for all } f \in \mathcal{H}ol(\mathbb{D}) \text{ and all } \varphi \in \mathrm{Aut}(\mathbb{D}).$$

We remark that Rubel and Timoney [36] proved that if X is any "reasonable" Möbius invariant space of analytic functions in $\mathbb{D}$, then it is continuously contained in the Bloch space. Let us be more precise.

Definition 2. Let X be a space of analytic functions in $\mathbb{D}$. A nonzero linear functional L on X is said to be *decent* if there exists a positive constant M and a compact subset K of $\mathbb{D}$ such that

$$|L(f)| \leq M \sup_{z \in K} |f(z)|, \quad \text{for all } f \in X.$$

We remark that for any given $a \in \mathbb{D}$, the point evaluation functional L_a defined by $L_a(f) = f(a)$ is a decent linear functional in $\mathcal{H}ol(\mathbb{D})$. Likewise, for $n = 1, 2, \dots$, the functional $L_{a,n}$ defined by $L_{a,n}(f) = f^{(n)}(a)$ is also a decent linear functional in $\mathcal{H}ol(\mathbb{D})$.

The above mentioned result of Rubel and Timoney is the following one.

Theorem A. *Let X be a Möbius-invariant linear space of analytic functions in $\mathbb{D}$ equipped with the Möbius-invariant seminorm ϱ. If there exists a decent linear functional L on X which is continuous with respect to ϱ, then $X \subset \mathcal{B}$ and there exists $A > 0$ such that*

$$\varrho_{\mathcal{B}}(f) \leq A\varrho(f), \quad \textit{for all } f \in X.$$

There are a lot of characterizations of the Bloch space. We shall see that some of them can be obtained using Theorem A. In order to do so we need some further results and concepts.

Lemma 2. *Let μ be a positive Borel measure in $\mathbb{D}$. Then the following assertions are equivalent:*
(i) *μ is a finite measure.*
(ii) *There is a positive constant C such that*

$$\int_{\mathbb{D}} (1 - |z|^2)^p |f'(z)|^p \, d\mu(z) \leq C \varrho_{\mathcal{B}}(f)^p, \quad \textit{for all } f \in \mathcal{B}$$

Proof. The implication (i) $\Rightarrow$ (ii) follows directly from the definition of $\mathcal{B}$.

Suppose (ii). As we mentioned above, there exist two functions $f_1, f_2 \in \mathcal{B}$ such that

$$(1 - |z|^2) \left(|f_1'(z)| + |f_2'(z)|\right) \geq 1, \quad \text{for all } z \in \mathbb{D}.$$

Then, we have

$$\begin{aligned}
\infty &> \int_{\mathbb{D}} (1 - |z|^2)^p |f_1'(z)|^p \, d\mu(z) + \int_{\mathbb{D}} (1 - |z|^2)^p |f_2'(z)|^p \, d\mu(z) \\
&= \int_{\mathbb{D}} (1 - |z|^2)^p \left(|f_1'(z)|^p + |f_2'(z)|^p\right) \, d\mu(z) \\
&\geq C_p \int_{\mathbb{D}} (1 - |z|^2)^p \left(|f_1'(z)| + |f_2'(z)|\right)^p \, d\mu(z) \\
&\geq C_p \mu\left(\mathbb{D}\right).
\end{aligned}$$

Hence, (i) holds. □

Definition 3. Suppose that $0<p<\infty$ and let μ be a positive Borel measure in $\mathbb{D}$. We let $\text{Möb}\,(D_p(\mu))$ denote the space of those functions $f\in\mathcal{H}ol(\mathbb{D})$ such that

$$\varrho_{\text{Möb}(D_p(\mu))}\stackrel{\text{def}}{=}\sup_{\varphi\in\text{Möb}(\mathbb{D})}\left(\int_{\mathbb{D}}(1-|z|^2)^p\left|(f\circ\varphi)'(z)\right|^p\,d\mu(z)\right)^{1/p}<\infty.$$

It is clear that $\text{Möb}\,(D_p(\mu))$ is a Möbius invariant space with the Möbius-invariant seminorm $\varrho_{\text{Möb}(D_p(\mu))}$.

Using Theorem A and Lemma 2 we obtain the following characterization of the Bloch space.

Theorem 9. *Suppose that $0<p<\infty$ and let μ be a positive and finite Borel measure in $\mathbb{D}$ such that there exists a decent linear functional L on $Möb\,(D_p(\mu))$ continuous with respect the seminorm $\varrho_{Möb(D_p(\mu))}$. Then,*

$$\mathcal{B}=Möb\,(D_p(\mu))\,,$$

and

$$\varrho_{Möb(D_p(\mu))}\asymp\varrho_{\mathcal{B}}(f),\quad f\in\mathcal{B}.$$

Theorem 9 includes as particular cases some well known results. Let us mention some of them.

Fix $\alpha>-1$ and $0<p<\infty$. Note that the weighted-area-measure A_α satisfies the conditions for the measure which appears in Theorem 9. Indeed, it is a finite measure and the functional L defined by $L(f)=f'(0)$ is a decent linear functional in $\mathcal{H}ol(\mathbb{D})$ and there exists a positive constant $C_{p,\alpha}$ such that

$$\begin{aligned}|L(f)|^p=|f'(0)|^p\leq& C_{p,\alpha}\int_0^1 r(1-r^2)^\alpha M_p^p(r,f')\,dr\\ \leq& C_{p,\alpha}\varrho^p_{\text{Möb}(D_p(A_\alpha))}(f),\end{aligned}$$

for all $f\in\text{Möb}\,(D_p(A_\alpha))$. Thus, Theorem 9 implies that

$$\mathcal{B}=\text{Möb}\,(D_p(A_\alpha))$$

and

$$\varrho^p_{\mathcal{B}}(f)\asymp\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}(1-|z|^2)^{p+\alpha}\left|(f\circ\varphi_a)'(z)\right|^p\,dA(z). \tag{3.1}$$

Taking $p=2$ in (3.1) we obtain that

$$\varrho_{\mathcal{B}}(f)\;\asymp\;\left(\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}(1-|z|^2)^s|\left(f\circ\varphi_a\right)'(z)|^2\,dA(z)\right)^{1/2},\ \text{for all } s>1.$$

This is equivalent to saying that

$$\mathcal{B} = Q_s, \quad s > 1,$$

a result which was proved by Xiao [45] for $s = 2$ and by Aulaskari and Lappan [6] for all $s > 1$. We refer to the monographs [46] and [47] for the theory of the Q_s-spaces.

Taking $\alpha = 0$ in (3.1), we easily obtain that

$$f \in \mathcal{B} \;\Leftrightarrow\; \left(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(1 - |\varphi_a(z)|^2\right)^2 (1 - |z|^2)^{p-2} |f'(z)|^p \, dA(z)\right)^{1/p},$$

a result which was proved by Stroethoff [40].

4. The Bloch space and hyperbolic geometry

The pseudohyperbolic metric in the unit disc will be denoted by δ:

$$\delta(z, w) = |\varphi_w(z)| = \left|\frac{z - w}{1 - \overline{w}z}\right|, \quad z, w \in \mathbb{D}.$$

The Schwarz-Pick lemma tells us that $\delta(f(z), f(w)) \le \delta(z, w)$ whenever f is an analytic self-map of $\mathbb{D}$ and z, $w \in \mathbb{D}$. Also, equality holds whenever f is a disc automorphism, *i.e.,* δ is invariant under the conformal maps of the disc onto itself.

The *pseudohyperbolic disc* of center a and radius r ($a \in \mathbb{D}$, $0 < r < 1$) is the set $\Delta(a, r) = \{z \in \mathbb{D} : \delta(a, z) < r\}$. It coincides with the Euclidean disc whose (Euclidean) radius R and center c are (see, e. g., p. 40 of [17]):

$$R = \frac{1 - |a|^2}{1 - r^2|a|^2} r, \qquad c = \frac{1 - r^2}{1 - r^2|a|^2} a.$$

A sequence $\{a_k\}_{k=1}^{\infty} \subset \mathbb{D}$ is said to be *uniformly discrete* if it is separated in the pseudohyperbolic metric, that is, if there exists a positive constant γ such that

$$\delta(a_j, a_k) \ge \gamma, \quad k = 1, 2, \ldots.$$

Uniformly discrete sequences play an important role in different and interesting questions of the theory of Bergman spaces, (see, e. g., [16], section 2.11 of [17], [18] and [23]).

An important subclass of uniformly discrete sequences is that of uniformly separated sequences or interpolating sequences. A sequence

$\{a_k\}$ of points in $\mathbb{D}$ is *uniformly separated* if there exists a positive constant γ such that

$$\prod_{j\neq k}\left|\frac{a_j-a_k}{1-\overline{a}_j a_k}\right| = \prod_{j\neq k}\delta(a_j,a_k)\geq\gamma, \quad k=1,2,\dots.$$

We recall that Carleson [11] (see also [15, Chapter 9]) proved that these are the universal interpolating sequences for the space H^∞ of bounded analytic functions in $\mathbb{D}$.

Definition 4. Suppose that $0<r<1$ and let $\{\lambda_k\}$ be a sequence of points in $\mathbb{D}$. We shall say that $\{\lambda_k\}$ is an *r-dense sequence* if $\mathbb{D}=\cup_k\Delta(\lambda_k,r)$. If in addition there exists a positive integer N such that no point $z\in\mathbb{D}$ belongs to more that N of the dilated discs $\Delta(\lambda_k,\frac{1+r}{2})$, we say that $\{\lambda_k\}$ is an *r-dense sequence of finite order*. The smallest N with this property will be called the order of $\{\lambda_k\}$.

The following result is stated as Lemma 12 on p. 62 of [17].

Proposition A. *For each $r\in(0,1)$ there exists a sequence of points in $\mathbb{D}$ which is an r-dense sequence of finite order.*

Using this and some other results one can prove the following:

Proposition 1. *Suppose that $0<r<1$ and let μ be a positive Borel measure in $\mathbb{D}$.*

(i) *Let $\alpha>1$. Then, μ is an α-Carleson measure if and only if there exists a constant $C_r>0$ such that*

$$\mu\left(\Delta(a,r)\right)\leq C_r(1-|a|)^\alpha, \quad a\in\mathbb{D}. \tag{4.1}$$

(ii) *Suppose that $s>0$ and μ is an s-logarithmic Carleson measure, that is, there exists a positive constant C such that*

$$\mu\left(S(I)\right)\leq\frac{C|I|}{\log^s\frac{2}{|I|}}, \quad \textit{for any interval } I\subset\partial\mathbb{D}. \tag{4.2}$$

Then there exists a positive constant C_r such that

$$\mu\left(\Delta(a,r)\right)\leq\frac{C_r(1-|a|)}{\log^s\frac{2}{1-|a|}}, \quad a\in\mathbb{D}. \tag{4.3}$$

Proposition 1 (i) for $\alpha=2$ is included in [17, Theorem 14, p. 62]. The argument used there in the proof (see [17, pp. 65-66]) can be used to prove (i) for any $\alpha>1$ and (ii) (see Lemma 9.5 of [20]). We omit the details.

Let β denote the *hyperbolic distance* in $\mathbb{D}$:

$$\beta(z,w) = \frac{1}{2}\log\frac{1+\delta(z,w)}{1-\delta(z,w)}.$$

Clearly, the hyperbolic metric is also Möbius invariant. We recall the following characterizations of Bloch functions (see, e. g. [10], [40] or [49, Chapter 5]).

Theorem B. *Let $0<r<1$. For a function $f\in\mathcal{H}ol(\mathbb{D})$ the following quantities are equivalent:*
(i) $\rho_{\mathcal{B}}(f)$.
(ii) $\varrho^{\star}(f) \stackrel{def}{=} \sup_{z\in\mathbb{D}}\sup_{w\in\Delta(z,r)}|f(z)-f(w)|$.
(iii) $\varrho^{\star,\star}(f) \stackrel{def}{=} \sup_{z,w\in\mathbb{D}}\frac{|f(z)-f(w)|}{\beta(z,w)}$.
Consequently,

$$f\in\mathcal{B} \quad\Leftrightarrow\quad \varrho^{\star}(f)<\infty \quad\Leftrightarrow\quad \varrho^{\star,\star}(f)<\infty.$$

Using these results and r-dense sequences of finite order we can give a "discrete formulation" of the concept p-Bloch-Carleson measures.

Proposition 2. *Suppose that $p\in(0,\infty)$, $r\in(0,1)$ and μ is a finite positive Borel measure. Let $\{\lambda_k\}\subset\mathbb{D}$ be an r-dense sequence of finite order. Then μ is a p-Bloch-Carleson measure if and only if there exists a positive constant $C=C(r,\mu)$ such that*

$$\sum_{k=1}^{\infty}|f(\lambda_k)|^p\mu\left(\Delta(\lambda_k,r)\right)\le C||f||_{\mathcal{B}}^p \tag{4.4}$$

for all $f\in\mathcal{B}$.

Proof. Suppose $f\in\mathcal{B}$ and $\{\lambda_k\}$ is an r-dense sequence of finite order.

If μ is a p-Bloch-Carleson measure in $\mathbb{D}$, then Theorem B gives

$$
\begin{aligned}
&\sum_{k=1}^{\infty} |f(\lambda_k)|^p \mu\left(\Delta(\lambda_k, r)\right) \\
&= \sum_{k=1}^{\infty} \int_{\Delta(\lambda_k,r)} |f(\lambda_k)|^p \, d\mu(z) \\
&\le C \left(\sum_{k=1}^{\infty} \int_{\Delta(\lambda_k,r)} |f(z) - f(\lambda_k)|^p \, d\mu(z) + \sum_{k=1}^{\infty} \int_{\Delta(\lambda_k,r)} |f(z)|^p \, d\mu(z) \right) \\
&\le C \left(\varrho^\star(f)^p \sum_{k=1}^{\infty} \mu\left(\Delta(\lambda_k, r)\right) + \sum_{k=1}^{\infty} \int_{\Delta(\lambda_k,r)} |f(z)|^p \, d\mu(z) \right) \\
&\le C \left(\varrho^\star(f)^p \mu\left(\mathbb{D}\right) + \int_{\mathbb{D}} |f(z)|^p \, d\mu(z) \right) \\
&\le C ||f||_{\mathcal{B}}^p.
\end{aligned}
$$

Conversely, if μ satisfies (4.4), bearing in mind again Theorem B, we deduce

$$
\begin{aligned}
&\int_{\mathbb{D}} |f(z)|^p \, d\mu(z) \\
&\le \sum_{k=1}^{\infty} \int_{\Delta(\lambda_k,r)} |f(z)|^p \, d\mu(z) \\
&\le C \left(\sum_{k=1}^{\infty} \int_{\Delta(\lambda_k,r)} |f(z) - f(\lambda_k)|^p \, d\mu(z) + \sum_{k=1}^{\infty} \int_{\Delta(\lambda_k,r)} |f(\lambda_k)|^p \, d\mu(z) \right) \\
&\le C \left(\varrho^\star(f)^p \sum_{k=1}^{\infty} \mu\left(\Delta(\lambda_k, r)\right) + \sum_{k=1}^{\infty} |f(\lambda_k)|^p \mu\left(\Delta(\lambda_k, r)\right) \right) \\
&\le C \left(\varrho^\star(f)^p \mu\left(\mathbb{D}\right) + ||f||_{\mathcal{B}}^p \right) \\
&\le C ||f||_{\mathcal{B}}^p.
\end{aligned}
$$

□

5. Special classes of p-Bloch-Carleson measures

In this section we shall prove Theorems 4 and 5 and some other related results. In connection with Proposition A, we shall prove next that a certain particular sequence is an r-dense sequence of finite order, for all r sufficiently close to 1. This sequence will be used in the proof of Theorem 4.

Lemma 3. *Let*

$$z_{k,j} \stackrel{def}{=} \left(1-2^{-k}\right)\exp\left(\frac{2\pi j}{2^k}i\right), \quad k=0,1,2\ldots \tag{5.1}$$
$$j=0,1,\ldots,2^k-1.$$

Then the following assertions hold:

(i) $\{z_{k,j}\}$ *is a uniformly discrete sequence with constant* $\gamma \geq \frac{1}{3}$.

(ii) $\{z_{k,j}\}$ *is an* r*-dense sequence of finite order for all* $r \geq r_0 \stackrel{def}{=} \frac{9}{\sqrt{82}}$.

(iii) *Let* $I_{k,j}$ *be the smallest arc on the circumference* $\{z \in \mathbb{D} : |z| = 1-2^{-k}\}$ *from* $z_{k,j}$ *to* $z_{k,j+1}$, *then* $\delta(z, z_{k,j}) \leq r_0$ *for all* $z \in I_{k,j}$.

Proof. The assertion in Part(i) follows from Theorem 3 in [18]. To prove (ii), for given $z = re^{i\theta} \in \mathbb{D}$, take $k \in \mathbb{N}$ and $j \in \{0,1,\ldots 2^k-1\}$ such that

$$1-2^{-k} \leq r < 1-2^{-(k+1)} \quad \text{and} \quad \theta \in \left[\frac{2\pi j}{2^k}, \frac{2\pi(j+1)}{2^k}\right).$$

Then, setting $r_k = 1-2^{-k}$, we have

$$\begin{aligned}
\frac{1}{1-\delta(z,z_{k,j})^2} &= \frac{(1-rr_k)^2 + 4rr_k\sin^2\left(\frac{\theta-\theta_{k,j}}{2}\right)}{(1-r^2)(1-r_k^2)} \\
&= 1 + \frac{(r-r_k)^2 + 4rr_k\sin^2\left(\frac{\theta-\theta_{k,j}}{2}\right)}{(1-r^2)(1-r_k^2)} \\
&\leq 1 + \frac{2^{-2k-2} + \pi^2 2^{-2k+2}}{2^{-k-1}2^{-k}} \\
&\leq 1 + 2^{-1} + 2^3\pi^2 \leq 82
\end{aligned}$$

that is, $\delta(z, z_{k,j}) \leq \frac{9}{\sqrt{82}}$, and therefore $\{z_{k,j}\}$ is an r-dense sequence for all $r \geq \frac{9}{\sqrt{82}}$.

Now, bearing in mind (i) and Lemma 1 of [18], we see that if $r \geq r_0$ and

$$N(r) \geq \frac{196}{(1-r)(3+r)},$$

then no point $z \in \mathbb{D}$ belongs to more that $N(r)$ of the dilated discs $\Delta\left(z_{k,j}, \frac{1+r}{2}\right)$. So we have (ii).

(iii) follows easily from the proof of (ii). This finishes the proof. $\square$

The following elementary lemma is well known (see, e. g. [17, Chapter 4].

Lemma A. *Let $0 < r < 1$. Then there exist positive constants C_1, C_2 C_3 and C_4 which depend only on r such that, if $a \in \mathbb{D}$ and $z \in \Delta(a,r)$, then*

$$\begin{aligned} 1-|a|^2 &\le C_1(1-|z|^2) \le C_2|1-\overline{a}z| \\ &\le C_3\, A\,(\Delta(a,r))^{1/2} \le C_4(1-|a|^2). \end{aligned} \tag{5.2}$$

Proof of Theorem 4. Let p, r, μ and A be as in Theorem 4. Theorem 1 (ii) shows that if μ is a p-Bloch-Carleson measure then it satisfies (1.3).

Suppose now that μ satisfies (1.3). Let $\{z_{k,j}\}$ be the sequence constructed in Lemma 3. Using Lemma 3 and Proposition 2, we see that it suffices to prove that

$$\sum_{k=0}^{\infty}\sum_{j=0}^{2^k-1} |f(z_{k,j})|^p \mu\,(\Delta(z_{k,j},r)) \le C||f||_{\mathcal{B}}^p, \quad f \in \mathcal{B}. \tag{5.3}$$

Hence, take $f \in \mathcal{B}$ and set

$$R_k = 1-2^{-k} \quad k = 0,1,\dots$$

Theorem B implies

$$\begin{aligned} M_p^p(R_k,f) &= \frac{1}{2\pi}\sum_{j=0}^{2^k-1}\int_{I_k,j} |f(R_ke^{it})|^p\,dt \\ &\ge C\sum_{j=0}^{2^k-1}|f(z_{j,k})|^p|I_{k,j}| - \varrho^\star(f)^p \\ &\ge C\frac{1}{2^k}\sum_{j=0}^{2^k-1}|f(z_{j,k})|^p - \varrho^\star(f)^p. \end{aligned} \tag{5.4}$$

Consequently, using (1.5), (5.4), Lemma A, Theorem B and (1.4), we obtain

(5.5)

$$\sum_{k=0}^{\infty}\sum_{j=0}^{2^k-1}|f(z_{k,j})|^p\mu\left(\Delta(z_{k,j},r)\right)$$
$$\leq C\sum_{k=0}^{\infty}\mu\left(\Delta(R_k,r)\right)\sum_{j=0}^{2^k-1}|f(z_{k,j})|^p$$
$$\leq C\sum_{k=0}^{\infty}\mu\left(\Delta(R_k,r)\right)2^k\left(M_p^p(R_k,f)+\varrho^\star(f)^p\right)$$
$$\leq C\left(\sum_{k=0}^{\infty}\mu\left(\Delta(R_k,r)\right)\sum_{j=0}^{2^k-1}M_p^p(R_k,f)+||f||_{\mathcal{B}}^p\sum_{k=0}^{\infty}\mu\left(\Delta(R_k,r)\right)2^k\right)$$
$$\leq C\left(\sum_{k=0}^{\infty}\sum_{j=0}^{2^k-1}M_p^p(R_k,f)\mu\left(\Delta(z_{k,j},r)\right)+||f||_{\mathcal{B}}^p\sum_{k=0}^{\infty}\sum_{j=0}^{2^k-1}\mu\left(\Delta(z_{k,j},r)\right)\right)$$
$$\leq C||f||_{\mathcal{B}}^p\left(\sum_{k=0}^{\infty}\sum_{j=0}^{2^k-1}\int_{\Delta(z_{k,j},r)}\left(\log\frac{e}{1-|z|}\right)^{p/2}d\mu(z)+\mu\left(\mathbb{D}\right)\right)$$
$$\leq C||f||_{\mathcal{B}}^p\left(\int_{\mathbb{D}}\left(\log\frac{e}{1-|z|}\right)^{p/2}d\mu(z)+\mu\left(\mathbb{D}\right)\right)$$
$$\leq C||f||_{\mathcal{B}}^p.$$

Hence, (5.3) holds. □

Now we turn to consider condition (1.2).

Proof of Theorem 5.

(i) ⇔ (ii) is obvious.

(ii) ⇒ (iii) follows from Theorem 1.

(iii) ⇒ (i). Set $f(z)=\log\frac{1}{1-z}$, $z\in\mathbb{D}$. Then $f\in\mathcal{B}$ and (iii) implies that $f\in L^p(\mu)$. Hence,

$$\infty>\int_{\mathbb{D}}\left|\log\frac{1}{1-z}\right|^p d\mu(z)=\sum_{k=1}^{\infty}a_k k^p.$$

This finishes the proof. □

As we mentioned in Section 1, the family of discrete measures μ constructed in Theorem 5 for which (ii) and (iii) are equivalent, are supported in a radius. It is natural to ask whether a condition "close" to this is needed to make such a construction. We shall show that the answer to this question is negative in a very strong sense.

Theorem 10. *Suppose that $0 < p < \infty$. For every $\eta > 1$ there is a sequence $\{z_k^{(\eta)}\}_{k=1}^{\infty} \subset \mathbb{D}$ having every point $\xi \in \partial\mathbb{D}$ as an accumulation point and with the property that if $\{a_k\}_{k=1}^{\infty}$ is a sequence of positive numbers and $\mu = \sum_{k=1}^{\infty} a_k \, \delta_{z_k^{(\eta)}}$, then the following statements are equivalent:*

(i) *μ satisfies (1.2).*
(ii) *μ is a p-Bloch-Carleson measure.*

The sequences $\{z_k^{(\eta)}\}$ which appears in the statement of Theorem 10 are interpolating sequences for the Bloch space, which, according to the equivalence (i) $\Leftrightarrow$ (iii) in Theorem B are defined as follows:

Definition 5. A sequence of points $\{z_n\} \subset \mathbb{D}$ is said to be an *interpolating sequence for the Bloch space* if for any sequence of complex numbers $\{w_n\}$ for which there exists a constant $C > 0$ such that

$$|w_n - w_m| \leq C\beta(z_n, z_m), \quad n, m = 1, 2 \dots,$$

one can find a function $f \in \mathcal{B}$ such that

$$f(z_n) = w_n, \quad n = 1, 2 \dots.$$

Boe and Nicolau [10] obtained the following characterization of interpolating sequences for the Bloch space (see also [31]).

Theorem C. *A sequence of points $\{z_n\} \subset \mathbb{D}$ is an interpolating sequence for the Bloch space if and only if the following two conditions hold:*

(a) *$\{z_n\}$ is the union of two uniformly discrete sequences.*
(b) *There exist constants $M > 0$ and $\alpha \in (0, 1)$ such that*

$$\#\left\{z_k \in S(I) : 2^{-N-1}|I| < 1 - |z_k| < 2^{-N}|I|\right\} \leq M 2^{\alpha n}, \; N = 1, 2, \dots,$$

for any interval $I \subset \partial\mathbb{D}$.

The sequence $\{z_{k,j}\}$ constructed in Lemma 3 is uniformly discrete but it is not an interpolating sequence for the Bloch space. Next, we modify this sequence to obtain an interpolating sequence for the Bloch space.

Lemma 4. *For $\eta > 1$ set*

$$z_{k,j}^{(\eta)} \stackrel{def}{=} \left(1 - 2^{-\eta k}\right) \exp\left(\frac{2\pi j}{2^k} i\right), \quad k = 0, 1, 2 \dots \qquad (5.6)$$

$$j = 0, 1, \dots, 2^k - 1.$$

The sequence $\{z_{k,j}^{(\eta)}\}$ *is an interpolating sequence for the Bloch space.*

Proof. Arguing as in the proof of Theorem 3 of [18] we can show that $\{z_{k,j}^{(\eta)}\}$ is a uniformly discrete sequence. Hence, it remains to prove that it satisfies condition (b) of Theorem C.

Let $I \subset \partial\mathbb{D}$ be an interval, and denote

$$S_k = \{z \in \mathbb{C} : |z| = 1 - 2^{-\eta k}\}, \quad k = 2, 3, \dots .$$

Let γ_k be the number of points of the sequence $\{z_{k,j}^{(\eta)}\}$ in the arc $S_k \cap S(I)$. It is easy to see that

$$\gamma_k \leq C\, 2^k |I|.$$

On the other hand,

$$\begin{aligned} &2^{-N-1}|I| < 1 - |z_{k,j}^{(\eta)}| < 2^{-N}|I| \\ &\Leftrightarrow \eta^{-1} \log_2 \left(\frac{2^N}{|I|}\right) < k < \eta^{-1} \log_2 \left(\frac{2^{N+1}}{|I|}\right). \end{aligned}$$

For simplicity, set $A_N = \eta^{-1} \log_2 \left(\frac{2^N}{|I|}\right)$, $B_N = \eta^{-1} \log_2 \left(\frac{2^{N+1}}{|I|}\right)$. Then it follows that

$$\begin{aligned} &\# \left\{ z_{k,j}^{(\eta)} \in S(I) : 2^{-N-1}|I| < 1 - |z_{k,j}^{(\eta)}| < 2^{-N}|I| \right\} \\ \leq &\sum_{A_N \leq k \leq B_N} \gamma_k \leq C|I| \sum_{A_N \leq k \leq B_N} 2^k \\ \leq &C\,|I| 2^{B_N} = C\,|I| \left(\frac{2^{N+1}}{|I|}\right)^{1/\eta} \leq C\, 2^{N/\eta} |I|^{1-\frac{1}{\eta}} \leq C\, 2^{N/\eta}. \end{aligned}$$

Since $\eta > 1$, this shows that $\{z_{k,j}^{(\eta)}\}$ satisfies condition (b) of Theorem C and finishes the proof. □

Lemma 5. *Suppose that* $\eta > 1$ *and let* $\{z_{k,j}^{(\eta)}\}$ *be the sequence defined in* (5.6)*. Then there exists a positive constant* $C = C(\eta)$ *such that*

$$\left| \log \frac{1}{1 - |z_{k,j}^{(\eta)}|} - \log \frac{1}{1 - |z_{n,l}^{(\eta)}|} \right| \leq C\beta \left(z_{k,j}^{(\eta)}, z_{n,l}^{(\eta)} \right) \tag{5.7}$$

for all k, n, j, l *with* $k, n \in \{0, 1, 2 \dots\}$, $j \in \{0, 1, \dots, 2^k - 1\}$ *and* $l \in \{0, 1, \dots, 2^n - 1\}$*.*

Proof. Without loss of generality we may assume that $k > n$. Then,

$$\left| \log \frac{1}{1 - |z_{k,j}^{(\eta)}|} - \log \frac{1}{1 - |z_{n,l}^{(\eta)}|} \right| = \log \frac{1 - |z_{n,l}^{(\eta)}|}{1 - |z_{k,j}^{(\eta)}|}.$$

So, using the elementary inequality $\frac{1+x^2}{1-x^2} \le \frac{1+x}{1-x}$ $(0<x<1)$, it is clear that it suffices to prove that there exists a positive constant $C=C(\eta)$ such that

$$\begin{aligned}\frac{1-|z_{n,l}^{(\eta)}|}{1-|z_{k,j}^{(\eta)}|} &\le \left(\frac{1+\varrho^2(z_{n,l}^{(\eta)}, z_{k,j}^{(\eta)})}{1-\varrho^2(z_{n,l}^{(\eta)}, z_{k,j}^{(\eta)})}\right)^C \\ &= \left(\frac{|1-\overline{z_{k,j}^{(\eta)}}z_{n,l}^{(\eta)}|^2+|z_{k,j}^{(\eta)}-z_{n,l}^{(\eta)}|^2}{(1-|z_{k,j}^{(\eta)}|^2)(1-|z_{n,l}^{(\eta)}|^2}\right)^C,\end{aligned} \tag{5.8}$$

whenever $k>n$, $j=0,1,\dots,2^k-1$, and $l=0,1,\dots,2^n-1$.

Now, take $C\in\mathbb{N}$ such that

$$\frac{C}{2}\left(1-\frac{1}{2^\eta}\right)>1. \tag{5.9}$$

Using (5.9) and the inequality $(1+b)^C \ge 1+Cb$, $(b\ge 0)$ we deduce that

$$\begin{aligned}&\left(\frac{|1-\overline{z_{k,j}^{(\eta)}}z_{n,l}^{(\eta)}|^2+|z_{k,j}^{(\eta)}-z_{n,l}^{(\eta)}|^2}{(1-|z_{k,j}^{(\eta)}|^2)(1-|z_{n,l}^{(\eta)}|^2)}\right)^C \\ &\ge \left(\frac{\left(1-|z_{k,j}^{(\eta)}||z_{n,l}^{(\eta)}|\right)^2+\left(|z_{k,j}^{(\eta)}|-|z_{n,l}^{(\eta)}|\right)^2}{(1-|z_{k,j}^{(\eta)}|^2)(1-|z_{n,l}^{(\eta)}|^2)}\right)^C \\ &= \left(1+2\frac{\left(|z_{k,j}^{(\eta)}|-|z_{n,l}^{(\eta)}|\right)^2}{(1-|z_{k,j}^{(\eta)}|^2)(1-|z_{n,l}^{(\eta)}|^2)}\right)^C \\ &\ge 1+2C\frac{\left(|z_{k,j}^{(\eta)}|-|z_{n,l}^{(\eta)}|\right)^2}{(1-|z_{k,j}^{(\eta)}|^2)(1-|z_{n,l}^{(\eta)}|^2)}, \\ &k>n,\ j=0,1,\dots,2^k-1,\ l=0,1,\dots,2^n-1.\end{aligned} \tag{5.10}$$

Putting together (5.8) and (5.10) we see that it suffices to prove that

$$\frac{1-|z_{n,l}^{(\eta)}|}{1-|z_{k,j}^{(\eta)}|} \le 1+2C\frac{\left(|z_{k,j}^{(\eta)}|-|z_{n,l}^{(\eta)}|\right)^2}{(1-|z_{k,j}^{(\eta)}|^2)(1-|z_{n,l}^{(\eta)}|^2)}, \tag{5.11}$$

whenever $k > n$, $j = 0, 1, \dots, 2^k - 1$ and $l = 0, 1, \dots, 2^n - 1$. Now,

$$(5.11) \Leftrightarrow \quad \frac{|z_{k,j}^{(\eta)}| - |z_{n,l}^{(\eta)}|}{1 - |z_{k,j}^{(\eta)}|} \leq 2C \frac{\left(|z_{k,j}^{(\eta)}| - |z_{n,l}^{(\eta)}|\right)^2}{(1 - |z_{k,j}^{(\eta)}|^2)(1 - |z_{n,l}^{(\eta)}|^2)}$$
$$\Leftrightarrow \quad (1 + |z_{k,j}^{(\eta)}|)(1 + |z_{n,l}^{(\eta)}|)(1 - |z_{n,l}^{(\eta)}|) \leq 2C \left(|z_{k,j}^{(\eta)}| - |z_{n,l}^{(\eta)}|\right).$$

Notice that (5.9) implies that the last inequality is true if $k > n$. Thus, (5.11) holds for the values of k, n, l, j under consideration. □

Theorem 10 is a consequence of the following result.

Proposition 3. *Suppose that $\eta > 1$ and let $\{z_{k,j}^{(\eta)}\}$ be the sequence defined in* (5.6). *Let $\{a_{k,j} : k = 0, 1, \dots,\ j = 0, 1, \dots 2^k - 1\}$ be a sequence of positive numbers and set*

$$\mu = \sum_{k=0}^{\infty} \sum_{j=0}^{2^k-1} a_{k,j} \delta_{z_{k,j}^{(\eta)}}.$$

Then the following conditions are equivalent:
(i) $\sum_{k=0}^{\infty} k^p \sum_{j=0}^{2^k-1} a_{k,j} < \infty$.
(ii) *μ satisfies* (1.2).
(iii) *μ is a p-Bloch-Carleson measure.*

Proof. We observe that

$$\int_{\mathbb{D}} \left(\log \frac{1}{1 - |z|}\right)^p d\mu(z) = \sum_{k=0}^{\infty} \sum_{j=0}^{2^k-1} a_{k,j} \left(\log \frac{1}{1 - |z_{k,j}^{(\eta)}|}\right)^p$$
$$= (\eta \log 2)^p \sum_{k=0}^{\infty} k^p \sum_{j=0}^{2^k-1} a_{k,j},$$

which proves (i) $\Leftrightarrow$ (ii).
(ii) $\Rightarrow$ (iii) follows from Theorem 1.
(iii) $\Rightarrow$ (ii). Denote

$$w_{k,j}^{(\eta)} = \log \frac{1}{1 - |z_{k,j}^{(\eta)}|} \qquad k, n = 0, 1, 2 \dots$$
$$j = 0, 1, \dots, 2^k - 1, \quad l = 0, 1, \dots, 2^n - 1.$$

Using Lemma 3 and Lemma 5 we see that there exists a function $F \in \mathcal{B}$ such that $F(z_{k,j}^{(\eta)}) = w_{k,j}^{(\eta)}$ for $k, n = 0, 1, 2 \dots$, $j = 0, 1, \dots, 2^k - 1$

and, $l = 0, 1, \dots, 2^n - 1$. Using (iii), we obtain

$$\begin{aligned}\infty > \int_{\mathbb{D}} |F(z)|^p \, d\mu(z) &= \sum_{k=0}^{\infty} \sum_{j=0}^{2^k-1} a_{k,j} \left| F\left(z_{k,j}^{(\eta)}\right) \right|^p \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{2^k-1} a_{k,j} \left(\log \frac{1}{1-|z_{k,j}^{(\eta)}|} \right)^p = (\eta \log 2)^p \sum_{k=0}^{\infty} k^p \sum_{j=0}^{2^k-1} a_{k,j},\end{aligned}$$

which proves (ii) and finishes the proof. □

6. Relationship between α-Carleson measures and p-Bloch-Carleson measures

In this section we shall prove Theorem 6. Its part (a) is a simple consequence of the following result.

Theorem 11. *Suppose that $0 < p < \infty$ and $s > p + 1$. Let μ be a positive Borel measure in $\mathbb{D}$ which is an s-logarithmic Carleson measure, then μ is a p-Bloch-Carleson measure.*

Proof. Take $r \in (0, 1)$ and let $\{\lambda_k\}_{k=1}^{\infty} \subset \mathbb{D}$ be an r-dense sequence of finite order. Using Lemma A and Proposition 1, we obtain

$$\begin{aligned}&\int_{\mathbb{D}} \left(\log \frac{e}{1-|z|} \right)^p d\mu(z) \\ \le &\sum_{k=1}^{\infty} \int_{\Delta(\lambda_k, r)} \left(\log \frac{e}{1-|z|} \right)^p d\mu(z) \\ \le &C \sum_{k=1}^{\infty} \left(\log \frac{e}{1-|\lambda_k|} \right)^p \mu\left(\Delta(\lambda_k, r)\right) \\ \le &C \sum_{k=1}^{\infty} \frac{\mu\left(\Delta(\lambda_k, r)\right)}{m\left(\Delta(\lambda_k, r)\right)} \int_{\Delta(\lambda_k, r)} \left(\log \frac{e}{1-|z|} \right)^p dA(z) \\ \le &C \sum_{k=1}^{\infty} \int_{\Delta(\lambda_k, r)} \frac{1}{1-|z|} \left(\log \frac{e}{1-|z|} \right)^{p-s} dA(z) \\ \le &C \int_{\mathbb{D}} \frac{1}{1-|z|} \left(\log \frac{e}{1-|z|} \right)^{p-s} dA(z) \\ < &\infty,\end{aligned}$$

which, using Theorem 1, implies that μ is a p-Bloch-Carleson measure.
□

Theorem 6 (b) for $2 < p < \infty$ follows from the constructions made in the proof of Theorem 2.1 of [21]. Indeed, let f and g be the functions constructed there. Then $f \in \mathcal{B}$ and the measure $\mu_{g,p}$ in $\mathbb{D}$ given by $d\mu_{g,p}(z) = (1-|z|^2)^{p-1}|g'(z)|^p\,dA(z)$ is a classical Carleson measure and satisfies

$$\int_{\mathbb{D}} |f(z)|^p\,d\mu_{g,p}(z) = \infty.$$

Hence $\mu_{g,p}$ is not a p-Bloch-Carleson measure. Next we shall give a proof of Theorem 6 (b) valid for all $p > 0$.

Proof of Theorem 6 (b). Take $p \in (0, \infty)$ and set

$$\lambda = \frac{1}{p}\left(1 + \frac{p}{2}\right).$$

Set $f(z) = \sum_{k=0}^{\infty} z^{2^k}$, $(z \in \mathbb{D})$. Then $f \in \mathcal{B}$ and it is well known that

$$M_2(r, f) \geq C\left(\log\frac{1}{1-r}\right)^{1/2}, \quad \frac{1}{2} < r < 1. \tag{6.1}$$

Since f is given by a power series with Hadamard gaps, using Theorem 8.25 in chapter V of Vol. I of [50], we see that there exist two absolute constants $A > 0$ and $B > 0$ such that for every $r \in (0, 1)$ the set

$$E_r = \{t \in [0, 2\pi] : |f(re^{it})| > BM_2(r, f)\}$$

has Lebesgue measure greater than or equal to A,

$$|E_r| \geq A, \quad 0 < r < 1. \tag{6.2}$$

Define

$$\phi(r) = \frac{1}{(1-r)\left[\log\left(\frac{e^\lambda}{1-r}\right)\right]^\lambda}, \quad 0 \leq r < 1. \tag{6.3}$$

Then ϕ is an increasing function defined in $(0, 1)$ and, since $\lambda p > 1$,

$$\int_0^1 (1-r)^{p-1}\phi^p(r)\,dr < \infty.$$

Using Theorem 3.3 and Theorem 3.2 of [21], we see that there exists a function $g \in \mathcal{H}ol(\mathbb{D})$ which is given by a power series with Hadamard gaps and such that the measure $\mu_{g,p}$ is a classical Carleson measure and satisfying

$$M_2(r, g') \geq \phi(r), \quad r \in (0, 1). \tag{6.4}$$

Now, we assert that there exists a positive constant C such

$$\int_{E_r} |g'(re^{it})|^p\,dt \geq CM_2^p(r, g'), \quad 0 < r < 1 \tag{6.5}$$

(see the last paragraph of p. 421 of [21] for the case $2 < p < \infty$, and Lemma 1 of [24] for the case $0 < p \le 2$). Bearing in mind the definition of the sets E_r $(0 < r < 1)$ and using (6.5), (6.1), (6.4) and the definition of λ, we obtain

$$\begin{aligned}
&\int_{\mathbb{D}} |f(z)|^p \, d\mu_{g,p}(z) \\
&= \int_{\mathbb{D}} (1-|z|^2)^{p-1} |g'(z)|^p |f(z)|^p \, dA(z) \\
&\ge C \int_{1/2}^{1} (1-r)^{p-1} \int_{E_r} |g'(re^{it})|^p |f(re^{it})|^p \, dt dr \\
&\ge C \int_{1/2}^{1} (1-r)^{p-1} M_2^p(r,f) \int_{E_r} |g'(re^{it})|^p \, dt dr \\
&\ge C \int_{1/2}^{1} (1-r)^{p-1} M_2^p(r,f) M_2^p(r,g') \, dr \\
&\ge C \int_{1/2}^{1} (1-r)^{p-1} \left(\log \frac{1}{1-r}\right)^{\frac{p}{2}} \phi^p(r) \, dr \\
&\ge C \int_{1/2}^{1} \frac{dr}{(1-r)\left(\log \frac{1}{1-r}\right)^{p\lambda - \frac{p}{2}}} \\
&= C \int_{1/2}^{1} \frac{dr}{(1-r)\left(\log \frac{1}{1-r}\right)} \\
&= \infty.
\end{aligned}$$

Since $f \in \mathcal{B}$, this implies that $\mu_{g,p}$ is not a p-Bloch-Carleson measure. □

Proof of Theorem 6 (c). Take $p > 0$ and $s > p+1$. Set

$$\mu = \sum_{k=1}^{\infty} \frac{1}{k^s} \delta_{1-e^{-k}}.$$

Using Theorem 5, we see that μ is a p-Bloch-Carleson measure.

For $0 < h < 1$, set $I_h = \{e^{it} : -\frac{h}{2} < t < \frac{h}{2}\}$. We have

$$\mu\left(S(I_h)\right) = \sum_{k \ge \log \frac{2\pi}{h}} \frac{1}{k^s} \asymp \left(\log \frac{1}{h}\right)^{1-s}. \tag{6.6}$$

Since $\frac{\left(\log \frac{1}{h}\right)^{1-s}}{h^\alpha} \to \infty$, as $h \to 0$, for every $\alpha > 0$, (6.6) implies that μ is not an α-Carleson measure for any $\alpha > 0$. □

7. THE CASE $p > 1$

Proof of Theorem 7. We shall use an argument which is closely related to that used by Arcozzi, Rochberg and Sawyer in the proof of [5, Theorem 7]. Let p, p' and μ be as in Theorem 7. Les us recall that $\mathcal{B} = (A^1)^\star$ under the pairing $\langle \cdot, \cdot \rangle_{A^2}$ defined by

$$\langle f, g\rangle_{A^2} = \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA(z)$$

(see [49, Theorem 5. 1. 4]). We also have $L^p(d\mu) = \big(L^{p'}(d\mu)\big)^\star$ under the pairing $\langle \cdot, \cdot \rangle_{L^2(d\mu)}$,

$$\langle f, g\rangle_{L^2(d\mu)} = \int_{\mathbb{D}} f(z)\overline{g(z)}\, d\mu(z).$$

Now, μ is a p-Bloch-Carleson measure if and only if $\mathcal{B} \subset L^p(d\mu)$ and, by the closed graph theorem, this is the same as saying that the inclusion map I_d is continuous from $\mathcal{B}$ to $L^p(d\mu)$. This is equivalent to saying that the map T defined by

$$\langle Tf, g\rangle_{A^2} = \langle f, g\rangle_{L^2(d\mu)} \tag{7.1}$$

is continuous from $L^{p'}(d\mu)$ to A^1. Summarizing, we have

(7.2)

$$\boxed{\text{The measure } \mu \text{ is a } p\text{-Bloch-Carleson measure}} \Leftrightarrow$$

$$\boxed{\text{the operator } T \text{ defined by (7.1) is continuous from } L^{p'}(d\mu) \text{ to } A^1.}$$

Now, since the Bergman kernel

$$K(z, w) = \frac{1}{(1 - z\overline{w})^2}, \quad z, w \in \mathbb{D},$$

reproduces each $f \in A^2$ (see, e. g., [49, p. 49]), (7.1) gives

$$\begin{aligned}\int_{\mathbb{D}} f(z)\overline{g(z)}\, d\mu(z) &= \int_{\mathbb{D}} f(z)\overline{\left(\int_{\mathbb{D}} \frac{g(w)}{(1 - \overline{w}z)^2}\, dA(w)\right)}\, d\mu(z)\\ &= \int_{\mathbb{D}} \overline{g(w)} \left(\int_{\mathbb{D}} \frac{f(z)}{(1 - \overline{z}w)^2}\, d\mu(z)\right)\, dA(z)\\ &= \int_{\mathbb{D}} \overline{g(w)}Tf(w)\, dA(w).\end{aligned}$$

Thus

$$Tf(w) = \int_{\mathbb{D}} \frac{f(z)}{(1 - \overline{z}w)^2}\, d\mu(z), \quad f \in A^2,\ w \in \mathbb{D}.$$

That is, $T = T_\mu$ and the theorem follows from (7.2). $\square$

Now we turn to work with weights ω which satisfy the Bekollé-Bonami $\mathcal{B}_{p,\alpha}$-condition. As mentioned in Section 1, this condition allows us to determine the dual space of the space $A^p(\omega)$. Indeed, we have (see [8] or [27]):

Theorem D. *Suppose that $p>1$, $\alpha>-1$ and ω is a weight which satisfies the Bekollé-Bonami $\mathcal{B}_{p,\alpha}$ condition. Then, the dual space of $A^p_\alpha(\omega)$ can be identified with $A^{p'}_\alpha\left(\omega^{1-p'}\right)$ under the pairing*

$$\langle f,g\rangle_{A^2_\alpha}=\int_{\mathbb{D}} f(z)\overline{g(z)}\,dA_\alpha(z)=\lim_{r\to1^-}\int_{\mathbb{D}} f(z)\overline{g(rz)}\,dA_\alpha(z). \tag{7.3}$$

This result implies the following one.

Lemma 6. *Suppose that $p>1$, $\alpha>-1$ and ω is a weight which satisfies the Bekollé-Bonami $\mathcal{B}_{p,\alpha}$ condition. Then, the dual space of $A^{p'}_\alpha\left(\omega^{1-p'}\right)$ can be identified with $A^p_\alpha(\omega)$ under the pairing defined in* (7.3).

Proof. Using the fact that $(1-p)(1-p')=1$, we easily deduce that ω satisfies the Bekollé-Bonami $\mathcal{B}_{p,\alpha}$ condition if and only if $\omega^{1-p'}$ satisfies the condition $\mathcal{B}_{p',\alpha}$.

Then, using Theorem D it follows that

$$\left(A^{p'}_\alpha\left(\omega^{1-p'}\right)\right)^\star=A^p_\alpha\left(\left(\omega^{1-p'}\right)^{1-p}\right)=A^p_\alpha(\omega)$$

under the A^2_α-pairing. This finishes the proof. □

We shall also need the following result (see [25, Theorem 1.21]).

Theorem E. *Suppose that $\alpha>-1$ and ω is a weight which satisfies the Bekollé-Bonami $\mathcal{B}_{p,\alpha}$ condition. Then, the dual space of A^1_α can be identified with $\mathcal{B}$ under the pairing defined in* (7.3).

Proof of Theorem 8. Using Lemma 6, Theorem E and writting $I_d=T^\star$ for an appropriate operator T, we have that

$$\begin{aligned}&\mathcal{B}\subset A^p_\alpha(\omega)\\ &\Leftrightarrow I_d:\mathcal{B}\to A^p_\alpha(\omega)\quad\text{is bounded}\\ &\Leftrightarrow T^\star:\left(A^1_\alpha\right)^\star\to\left(A^{p'}_\alpha\left(\omega^{1-p'}\right)\right)^\star\quad\text{is bounded}\\ &\Leftrightarrow T:A^{p'}_\alpha\left(\omega^{1-p'}\right)\to A^1_\alpha\quad\text{is bounded,}\end{aligned}$$

where T is determined by the identity

$$\langle T(f),g\rangle_{A^2_\alpha}=\langle f,T^\star(g)\rangle_{A^2_\alpha}=\langle f,g\rangle_{A^2_\alpha}.$$

It follows that $T = I_d$, that is,

$$\mathcal{B} \subset A^p_\alpha(\omega) \Leftrightarrow A^{p'}_\alpha\left(\omega^{1-p'}\right) \subset A^1_\alpha.$$

Let us prove that this last inclusion holds. Let $f \in A^{p'}_\alpha\left(\omega^{1-p'}\right)$, then by Hölder's inequality

$$\begin{aligned}\int_{\mathbb{D}} |f(z)|\, dA_\alpha(z) &= \int_{\mathbb{D}} |f(z)|\omega^{\frac{1-p'}{p'}}(z)\omega^{\frac{p'-1}{p'}}(z)\, dA_\alpha(z)\\ &= \int_{\mathbb{D}} |f(z)|\omega^{\frac{1-p'}{p'}}(z)\omega^{\frac{1}{p}}(z)\, dA_\alpha(z)\\ &\le ||f||^{p'}_{A^{p'}_\alpha(\omega^{1-p'})}\left(\int_{\mathbb{D}} \omega(z)\, dA_\alpha(z)\right)^{1/p} < \infty,\end{aligned}$$

since ω satisfies the Bekollé-Bonami $\mathcal{B}_{p,\alpha}$ condition. This finishes the proof. □

8. Some results on the space $H^\infty_{\log}$

Our main objective in this section is to give a proof of Theorem 3. In order to do so we shall start proving a number of results on the space $H^\infty_{\log}$ which may be of independent interest.

The following result is a direct consequence of Cauchy's integral formula.

Proposition 4. *If $f(z) = \sum_{k=0}^\infty a_k z^k \in H^\infty_{\log}$, then*

$$\sup_{k\ge 2} \frac{|a_k|}{\log k} < \infty. \tag{8.1}$$

We observe that (8.1) is not sufficient to assert that $f \in H^\infty_{\log}$, even for $f \in \mathcal{L}$, as the example $f(z) = \sum_{k=1}^\infty k^{1/2} z^{2^k}$ shows. However, we shall prove that condition (8.1) does imply that $f \in H^\infty_{\log}$ if f belongs to a certain subclass of $\mathcal{L}$.

Theorem 12. *Let $f \in \mathcal{H}ol(\mathbb{D})$ with the power series expansion $f(z) = \sum_{k=0}^\infty a_k z^{n_k}$ and suppose that there exist $\lambda > 1$ and $A > 1$ such that*

$$n_k^\lambda \le n_{k+1} \le n_k^{A\lambda}, \quad \text{for all } k. \tag{8.2}$$

Then, the following conditions are equivalent
- (i) $f \in H^\infty_{\log}$.
- (ii) $\sup_{k\ge 0} \frac{|a_k|}{\log n_k} < \infty$.

(i) $\Rightarrow$ (ii) follows by Proposition 4.

Proof of (ii)$\Rightarrow$ (i). Take $\lambda > 1$ and $A > 1$ such that (8.2) holds. Let us assume, without loss of generality, that $n_0 \ge 1$. It is clear that it suffices to prove that there exists a positive constant $C = C(\lambda)$, such that

$$\sum_{k=1}^{\infty}(\log n_k) r^{n_k} \le C \log \frac{e}{1-r}, \quad 0 \le r < 1. \tag{8.3}$$

We recall that there exists $C = C(\lambda) > 0$ such that

$$\sum_{k=0}^{\infty} r^{\lambda^k} \le C \log \frac{e}{1-r}, \quad 0 \le r < 1. \tag{8.4}$$

It follows from (8.2) that

$$\log n_{k+1} - \log n_k \ge (\lambda - 1) \log n_k, \quad k \in \mathbb{N}, \tag{8.5}$$

and

$$\frac{\log n_{k+1}}{\log n_k} \le A\lambda, \quad k \in \mathbb{N}, \tag{8.6}$$

For $k = 0, 1, 2, ...,$ set $I(k) = [n_k, n_{k+1})$ and let β_k denote the number of terms of the sequence $\{\lambda^j n_0\}_{j=0}^{\infty}$ which are contained in $I(k)$. Then, we claim that

$$\log n_{k+1} - \log n_k \le (\beta_k + 1) \log \lambda. \tag{8.7}$$

Indeed, let j_k and $j_k^\star$, be the smallest and the biggest non-negative integer j such that $\lambda^j n_0 \in I(k)$. That is,

$$\lambda^{j_k - 1} n_0 < n_k \le \lambda^{j_k} n_0 \le \lambda^{j_k + 1} n_0 \le \dots \lambda^{j_k^\star} n_0 < n_{k+1} \le \lambda^{j_k^\star + 1} n_0.$$

Notice that $\beta_k = j_k^\star + 1 - j_k$ and

$$\begin{aligned} \log n_{k+1} - \log n_k &\le \log\left(\lambda^{j_k^\star + 1} n_0\right) - \log\left(\lambda^{j_k - 1} n_0\right) \\ &= (j_k^\star + 2 - j_k) \log \lambda \\ &= (\beta_k + 1) \log \lambda, \end{aligned}$$

as claimed.

Putting together (8.6), (8.5), (8.7) and (8.4), we deduce that

$$\begin{aligned}\sum_{k=1}^{\infty} \log n_k r^{n_k} &= \sum_{k=0}^{\infty} \log n_{k+1} r^{n_{k+1}}\\ &\le A\lambda \sum_{k=0}^{\infty} \log n_k r^{n_{k+1}}\\ &\le \frac{A\lambda}{\lambda-1} \sum_{k=0}^{\infty} \left(\log n_{k+1} - \log n_k\right) r^{n_{k+1}}\\ &\le \frac{A\lambda \log\lambda}{\lambda-1} \left(\sum_{k=0}^{\infty} \beta_k r^{n_{k+1}} + \sum_{k=0}^{\infty} r^{n_{k+1}} \right)\\ &\le C \left(\sum_{k=0}^{\infty} r^{\lambda^k n_0} + \sum_{k=0}^{\infty} r^{n_{k+1}} \right)\\ &\le C \log \frac{e}{1-r},\end{aligned}$$

which gives (8.3) and finishes the proof. □

We can now prove Theorem 2 with an argument similar to that used by Ramey and Ullrich in the proof of Proposition 5.4 of [35].

Proof of Theorem 2. Let $f(z) = \sum_{j=0}^{\infty} q^j z^{q^{q^j}}$, where q is a large positive integer to be determined. Theorem 12 implies that $f \in H^\infty_{\log}$. We first show that

$$|f(z)| \ge C \log \frac{1}{1-|z|}, \quad \text{if } 1 - q^{-q^k} \le |z| \le 1 - q^{-q^{k+\frac{1}{2}}}. \tag{8.8}$$

We have that

$$\begin{aligned}|f(z)| &\ge q^k |z|^{q^{q^k}} - \sum_{j=0}^{k-1} q^j |z|^{q^{q^j}} - \sum_{j=k+1}^{\infty} q^j |z|^{q^{q^j}}\\ &= I(z) - II(z) - III(z), \quad z \in \mathbb{D}.\end{aligned} \tag{8.9}$$

Take a point z as in (8.8) and let $x = |z|^{q^{q^k}}$. Then, if q is sufficiently large

$$\frac{1}{3} \le x \le \left(\frac{1}{2}\right)^{q^{\left(q^k - q^{k+\frac{1}{2}}\right)}}. \tag{8.10}$$

It follows that

$$I(z) \ge \frac{q^k}{3} \tag{8.11}$$

and

$$II(z) \leq \sum_{j=0}^{k-1} q^j \leq \frac{q^k}{q-1}. \tag{8.12}$$

In order to estimate $III(z)$ we observe that if $j \geq k+1$,

$$q^{q^{j+1}} - q^{q^j} \geq q^{q^{k+2}} - q^{q^{k+1}}.$$

Then, bearing in mind (8.10) we have that

$$\begin{aligned} III(z) &\leq q^{k+1}|z|^{q^{q^{k+1}}} \sum_{j=0}^{\infty} \left[q|z|^{\left(q^{q^{k+2}} - q^{q^{k+1}}\right)} \right]^j \\ &\leq \frac{q^{k+1}|z|^{q^{q^{k+1}}}}{1 - q|z|^{\left[q^{q^{k+2}} - q^{q^{k+1}}\right]}} \\ &\leq q^k \frac{q x^{q^{\left(q^{k+1}-q^k\right)}}}{1 - q x^{\left[q^{\left(q^{k+2}-q^k\right)} - q^{\left(q^{k+1}-q^k\right)}\right]}} \\ &\leq q^k \frac{q\left(\frac{1}{2}\right)^{q^{\left(q^{k+1}-q^{k+\frac{1}{2}}\right)}}}{1 - q\left(\frac{1}{2}\right)^{\left[q^{\left(q^{k+2}-q^{k+\frac{1}{2}}\right)} - q^{\left(q^{k+1}-q^{k+\frac{1}{2}}\right)}\right]}}. \end{aligned} \tag{8.13}$$

Since, $\lim_{q\to\infty}\left(q^{k+1} - q^{k+\frac{1}{2}}\right) = \infty$, we see that by taking q sufficiently large, (8.11), (8.12) and (8.13) give

$$|f(z)| \geq \frac{q^k}{4} \geq \frac{1}{4q^{\frac{1}{2}}\log q} \log \frac{1}{1-|z|}, \quad \text{if } 1 - q^{-q^k} \leq |z| \leq 1 - q^{-q^{k+\frac{1}{2}}}.$$

In a similar way it can be proved that if $g(z) = \sum_{j=0}^{\infty} q^{j+\frac{1}{2}} z^{n_j}$ where n_j is the integer closest to $q^{j+\frac{1}{2}}$, then

$$|g(z)| \geq C_1 \log \frac{1}{1-|z|}, \quad \text{if } 1 - q^{-q^{k+\frac{1}{2}}} \leq |z| \leq 1 - q^{-q^{k+1}}.$$

Now the theorem follows by taking

$$f_1(z) = A + B\, f(z) \quad \text{and} \quad f_2(z) = C\, g(z), \quad (z \in \mathbb{D}),$$

for some appropriate positive constants A, B, C. □

Proof of Theorem 3. We have already noticed that if μ satisfies (1.2) then it is a p-Carleson measure for the space $H^\infty_{\log}$ (see Section 1).

Suppose now that μ is p-Carleson measure for $H^\infty_{\log}$. Using Theorem 2, we can take $f_1, f_2 \in H^\infty_{\log}$ satisfying

$$|f_1(z)| + |f_2(z)| \geq \log \frac{1}{1-|z|}, \quad z \in \mathbb{D}.$$

Using this and the simple fact that $(a+b)^p \leq 2^p(a^p + b^p)$, whenever $a, b \geq 0$, we deduce that

$$\begin{aligned}\int_{\mathbb{D}} \left(\log \frac{1}{1-|z|}\right)^p d\mu(z) &\leq \int_{\mathbb{D}} \left(|f_1(z)| + |f_2(z)|\right)^p d\mu(z) \\ &\leq C \left(\int_{\mathbb{D}} |f_1(z)|^p \, d\mu(z) + \int_{\mathbb{D}} |f_2(z)|^p \, d\mu(z)\right) \\ &< \infty.\end{aligned}$$

□

9. Operators acting on $H^\infty_{\log}$

For g analytic in $\mathbb{D}$, the multiplication operator M_g is defined by

$$M_g(f)(z) \stackrel{\text{def}}{=} g(z)f(z), \quad f \in \mathcal{H}ol(\mathbb{D}),\ z \in \mathbb{D}. \tag{9.1}$$

If X and Y are two spaces of analytic functions in $\mathbb{D}$, we let $M(X, Y)$ denote the space of all multipliers from X to Y, i e.,

$$M(X, Y) = \{g \in \mathcal{H}ol(\mathbb{D}) : M_g(X) \subset Y\}.$$

If φ is an analytic self-map of $\mathbb{D}$, the composition operator C_φ is defined by

$$C_\varphi f = f \circ \varphi, \quad f \in \mathcal{H}ol(\mathbb{D}). \tag{9.2}$$

More generally, given φ an analytic self-map of $\mathbb{D}$ and u an analytic function in $\mathbb{D}$, the weighted composition operator C^u_φ with symbols u and φ is defined as follows

$$C^u_\varphi(f)(z) = u(z)(f \circ \varphi)(z), \quad f \in \mathcal{H}ol(\mathbb{D}),\ z \in \mathbb{D}. \tag{9.3}$$

We remark that the weighted composition operators contain as particular cases the multiplication operators (taking φ equal to the identity mapping) and the composition operators (taking $u \equiv 1$).

There is a very extense literature on weighted composition operators. Les us just mention that two excelent monographs on composition operators are those of Shapiro [37] and Cowen and MacCluer [13].

The question of boundedness and compactness of these operators has been studied in many function spaces. We shall use the results obtained in Section 8 to study the boundedness and compactness of

C_φ^u from $H^\infty_{\log}$ to the weighted Bergman spaces A^p_α. We shall prove the following result.

Theorem 13. *Let φ be an analytic self-map of $\mathbb{D}$, u an analytic function in $\mathbb{D}$, $p > 0$ and $\alpha > -1$. Then the following assertions are equivalent:*

(1) *C_φ^u is a continuous operator from $H^\infty_{\log}$ to A^p_α.*

(2) $\displaystyle\int_{\mathbb{D}} |u(z)|^p \left(\log \frac{e}{1-|\varphi(z)|}\right)^p \, dA_\alpha(z) < \infty.$

(3) $\displaystyle\lim_{r\to 1} \int_{|\varphi(z)|>r} |u(z)|^p \left(\log \frac{e}{1-|\varphi(z)|}\right)^p \, dA_\alpha(z) = 0.$

(4) *C_φ^u is a compact operator from $H^\infty_{\log}$ to A^p_α.*

We shall prove first that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

Notice that the implication (2) $\Rightarrow$ (3) follows from the dominated convergence theorem, while the implication (3) $\Rightarrow$ (2) is obvious. The implication (2) $\Rightarrow$ (1) follows directly from the definition of the space $H^\infty_{\log}$.

Proof of (1) $\Rightarrow$ (2). Suppose (1). We use Theorem 2 to pick two functions $f_1, f_2 \in H^\infty_{\log}$ such that

$$|f_1(z)| + |f_2(z)| \geq \log \frac{1}{1-|z|}, \quad z \in \mathbb{D}. \tag{9.4}$$

Now, (1) implies that

$$\int_{\mathbb{D}} |u(z)|^p \, |(f_j \circ \varphi)(z)|^p \, dA_\alpha(z) < \infty, \quad j = 1, 2. \tag{9.5}$$

Using (9.4), the elementary inequality $(a+b)^p \leq 2^p(a^p + b^p)$ $(a, b \geq 0)$ and (9.5), we obtain

$$\begin{aligned}
&\int_{\mathbb{D}} |u(z)|^p \left(\log \frac{1}{1-|\varphi(z)|}\right)^p \, dA_\alpha(z) \\
&\leq \int_{\mathbb{D}} |u(z)|^p \left(|(f_1 \circ \varphi)(z)| + |(f_2 \circ \varphi)(z)|\right)^p \, dA_\alpha(z) \\
&\leq C_p \int_{\mathbb{D}} |u(z)|^p \left(|(f_1 \circ \varphi)(z)|^p + |(f_2 \circ \varphi)(z)|^p\right) \, dA_\alpha(z) \\
&< \infty. \quad \Box
\end{aligned}$$

The implication (4) $\Rightarrow$ (1) is obvious. Hence, it only remains to prove that (3) $\Rightarrow$ (4). We shall also use the following result of Tjani [41, Lemma 3.7].

Lemma B. *Let X and Y be two Banach spaces (or complete metric spaces) of analytic functions on $\mathbb{D}$, and let $T : X \to Y$ be a linear operator. Suppose that the following conditions are satisfied:*

(a) *The point evaluation functionals on Y are bounded.*
(b) *For every bounded sequence in X, there is a subsequence which converges uniformly to an element of X on compact subsets of $\mathbb{D}$.*
(c) *If $\{f_n\} \subset X$ converges uniformly to zero on compact subsets of $\mathbb{D}$, then $\{T(f_n)\}$ converges uniformly to zero on compact subsets of $\mathbb{D}$.*

Then T is a compact operator from X to Y if and only if for any bounded sequence $\{f_n\}$ in X such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, the sequence $\{T(f_n)\}$ converges to zero in the norm (or in the metric) of Y.

Proof of the implication (3) $\Rightarrow$ (4) *in Theorem 13.*
Suppose (3). It is clear that we are in the conditions of Lemma B. Hence, it suffices to show that if $\{f_n\}$ is a bounded sequence in $H^\infty_{\log}$ such that $f_n \to 0$, as $n \to \infty$, uniformly on compact subsets of $\mathbb{D}$, then

$$\|C^u_\varphi(f_n)\|_{A^p_\alpha} \to 0, \text{ as } n \to \infty.$$

Hence, take such a sequence $\{f_n\}$. We have

$$\begin{aligned}
&\int_{\{|\varphi(z)|>r\}} |C^u_\varphi(f_n)(z)|^p \, dA_\alpha(z) \\
&\leq ||f_n||^p_{H^\infty_{\log}} \int_{\{|\varphi(z)|>r\}} |u(z)|^p \left(\log \frac{e}{1-|\varphi(z)|}\right)^p dA_\alpha(z), \\
&\leq C \int_{\{|\varphi(z)|>r\}} |u(z)|^p \left(\log \frac{e}{1-|\varphi(z)|}\right)^p dA_\alpha(z), \quad \text{for all } n.
\end{aligned}$$

Take $\varepsilon > 0$. The last inequality and (3) imply that there exists $r_0 \in (0,1)$ such that

$$\int_{\{|\varphi(z)|>r_0\}} |C^u_\varphi(f_n)(z)|^p \, dA_\alpha(z) < \frac{\varepsilon}{2}, \quad \text{for all } n.$$

On the other hand, since f_n converges to zero uniformly on compact subsets of $\mathbb{D}$, there exists $N \in \mathbb{N}$ such that

$$\int_{\{|\varphi(z)|\leq r_0\}} |C^u_\varphi(f_n)(z)|^p \, dA_\alpha(z) \quad \leq \quad \frac{\varepsilon}{2}, \quad \text{for all } n \geq N.$$

Then, it follows that $\|uC_\varphi(f_n)\|_{A^p_\alpha} \to 0$ as $n \to \infty$. This finishes the proof. $\square$

As we mentioned above composition operators are particular cases of weighted composition operators. Consequently, we can state the following result.

Corollary 3. *Let φ be an analytic self-map of $\mathbb{D}$, $p>0$ and $\alpha>-1$. Then the following assertions are equivalent:*

(1) *C_φ is a continuous operator from $H^\infty_{\log}$ to A^p_α.*
(2) $\displaystyle\int_{\mathbb{D}}\left(\log\frac{e}{1-|\varphi(z)|}\right)^p dA_\alpha(z)<\infty.$
(3) $\displaystyle\lim_{r\to 1}\int_{|\varphi(z)|>r}\left(\log\frac{e}{1-|\varphi(z)|}\right)^p dA_\alpha(z)=0.$
(4) *C_φ is a compact operator from $H^\infty_{\log}$ to A^p_α.*

A similar statement is valid for multiplication operators. However, we shall obtain some further results for these operators as well as for a class of integration operators.

Given $g\in\mathcal{H}ol(\mathbb{D})$ integration operator T_g is defined as follows:

$$T_g(f)(z)=\int_0^z f(\xi)g'(\xi)\,d\xi,\quad f\in\mathcal{H}ol(\mathbb{D}),\ z\in\mathbb{D}. \tag{9.6}$$

The operators T_g have been studied in a number of papers and contain as special cases a number of important operators such as the integration operator (T_g with $g(z)=z$) and the Cesàro operator (T_g with $g(z)=\log(1/(1-z))$). Let us mention that Pommerenke [34] proved that T_g is bounded in the Hardy space H^2 if and only if $g\in BMOA$, Aleman and Cima characterized in [1] those $g\in\mathcal{H}ol(\mathbb{D})$ for which T_g maps H^p into H^q, Aleman and Siskakis [2] studies the operators T_g acting on Bergman spaces and the first two authors of this paper studied them acting on spaces of Dirichlet type in [22].

We have the following result.

Theorem 14. *Suppose that $0<p<\infty$ and $-1<\alpha<\infty$ and let $g\in\mathcal{H}ol(\mathbb{D})$. Then the following conditions are equivalent*

(1) $g\in M\left(H^\infty_{\log},A^p_\alpha\right)$.
(2) $\int_{\mathbb{D}}|g(z)|^p\left(\log\frac{e}{1-|z|}\right)^p(1-|z|)^\alpha\,dA(z)<\infty.$
(3) $\int_{\mathbb{D}}|g'(z)|^p\left(\log\frac{e}{1-|z|}\right)^p(1-|z|)^{\alpha+p}\,dA(z)<\infty.$
(4) *T_g is a bounded operator from $H^\infty_{\log}$ to A^p_α.*
(5) *The multiplication operator M_g is a compact operator from $H^\infty_{\log}$ to A^p_α.*
(6) *The operator T_g is a compact operator from $H^\infty_{\log}$ to A^p_α.*

Proof. Theorem 13 shows that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (5). The equivalence (2) $\Leftrightarrow$ (3) follows from results of Siskakis (see Theorem 1.1 and Example 3.1 of [38]) and of Pavlović and Peláez [32, Theorem 1.1].

A result of Hardy and Littlewood asserts that for $0 < p < \infty$ and $\alpha > -1$

$$\int_{\mathbb{D}} |g(z)|^p (1-|z|)^\alpha \, dA(z) \asymp |f(0)|^p + \int_{\mathbb{D}} |g'(z)|^p (1-|z|)^{\alpha+p} \, dA(z)$$

for all $g \in \mathcal{H}ol(\mathbb{D})$ (see Chapter 5 of [15] for information and references and Theorem 6 of [19] for a proof). Using this we have

$$\begin{aligned} & T_g(H^\infty_{\log}) \subset A^p_\alpha \\ & \Leftrightarrow \int_{\mathbb{D}} |(1-|z|)^{\alpha+p} |f(z) g'(z)|^p \, dA(z) < \infty, \text{for all } f \in H^\infty_{\log} \\ & \Leftrightarrow (1-|z|)^{\alpha+p} |g'(z)|^p \, dA(z) \text{ is a } p - H^\infty_{\log}\text{-Carleson measure} \\ & \Leftrightarrow \int_{\mathbb{D}} |g'(z)|^p (1-|z|)^{\alpha+p} \left(\log \frac{e}{1-|z|}\right)^p dA(z) < \infty. \end{aligned}$$

This shows that (3) $\Leftrightarrow$ (4).

The implication (6) $\Rightarrow$ (4) is obvious. Thus, it only remains to prove that (3) $\Rightarrow$ (6).

Proof of (3)$\Rightarrow$ (6). The proof is similar to that of the implication (3) $\Rightarrow$ (4) in Theorem 13.

Suppose (3). By Lemma B, it suffices to show that if $\{f_n\}$ is a bounded sequence in $H^\infty_{\log}$ such that $f_n \to 0$, as $n \to \infty$, uniformly on compact subsets of $\mathbb{D}$, then

$$\|T_g(f_n)\|_{A^p_\alpha} \to 0, \text{ as } n \to \infty. \tag{9.7}$$

Now, using the above mentioned result of Hardy and Littlewood, this is equivalent to saying that

$$\int_{\mathbb{D}} |f_n(z) g'(z)|^p (1-|z|)^{\alpha+p} \, dA(z) \to 0, \quad \text{as } n \to \infty. \tag{9.8}$$

Hence, take such a sequence $\{f_n\}$. Notice that, by the dominated convergence theorem, (iii) implies

$$\lim_{r\to 1} \int_{\{|z|>r\}} |g'(z)|^p \left(\log \frac{e}{1-|z|}\right)^p (1-|z|)^{\alpha+p} \, dA(z) = 0. \tag{9.9}$$

We have

$$\begin{aligned}
&\int_{\{|z|>r\}} |f_n(z)g'(z)|^p(1-|z|)^{\alpha+p}\,dA(z)\\
&\le ||f_n||^p_{H^\infty_{\log}} \int_{\{|z|>r\}} |g'(z)|^p \left(\log\frac{e}{1-|z|}\right)^p (1-|z|)^{\alpha+p}\,dA(z)\\
&\le C \int_{\{|z|>r\}} |g'(z)|^p \left(\log\frac{e}{1-|z|}\right)^p (1-|z|)^{\alpha+p}\,dA(z), \quad \text{for all } n.
\end{aligned}$$

Take $\varepsilon > 0$. The last inequality and (9.9) imply that there exists $r_0 \in (0,1)$ such that

$$\int_{\{|z|>r_0\}} |f_n(z)g'(z)|^p(1-|z|)^{\alpha+p}\,dA(z) < \frac{\varepsilon}{2}, \quad \text{for all } n.$$

On the other hand, since f_n converges to zero uniformly on compact subsets of $\mathbb{D}$, there exists $N \in \mathbb{N}$ such that

$$\int_{\{|z|\le r_0\}} |f_n(z)g'(z)|^p(1-|z|)^{\alpha+p}\,dA(z) \quad \le \quad \frac{\varepsilon}{2}, \quad \text{for all } n \ge N.$$

Then, (9.8) follows. This finishes the proof. □

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