Univalent functions in Hardy, Bergman, Bloch and related spaces

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Abstract

The aim of this paper is to show that univalent functions in several classical function spaces can be characterized by integral conditions involving the maximum modulus function. For a suitable choice of parameters the established condition or its appropriate variant reduces to a known characterization of univalent functions in the Hardy or the weighted Bergman space, and gives a new characterization of univalent functions in several Möbius invariant function spaces such as $BMOA$, Q_p or the Bloch space. It is proved, for example, that univalent functions in the Dirichlet type space $\mathcal{D}^p_{p+\alpha}$ are the same as the univalent functions in H^p_α and S^p_α if $p \geq 2$. Moreover, it is shown that there is in a sense a much smaller Möbius invariant subspace of the Bloch space than Q_p still containing all univalent Bloch functions.

1. Introduction

The class $\mathcal{U}$ of univalent (analytic and one-to-one) functions in the unit disc has been extensively studied in the mathematical literature since the beginning of the last century. This study reflects the close connection between geometric and analytic properties that coexist in the function theory. After the result of Prawitz (1927), which implies that any univalent function belongs to the Hardy space H^p for all $p < \frac{1}{2}$, univalent functions in different spaces of analytic functions in the unit disc have been characterized by several authors. The description of univalent functions in the Hardy space H^p follows by the works due to Prawitz (1927),

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Hardy and Littlewood (1932) and Pommerenke (1977), while univalent functions in the weighted Bergman space A^p_α were recently characterized by Baernstein, Girela and Peláez (2004).

The aim of the present paper is to show that univalent functions in several classical function spaces can be characterized by integral conditions involving the maximum modulus function. For a suitable choice of parameters the established condition or its appropriate variant reduces to a known characterization of univalent functions in the Hardy or the weighted Bergman space, and gives a new characterization of univalent functions in several Möbius invariant function spaces such as $BMOA$, Q_p or the Bloch space.

Theorem 1 shows that the Dirichlet type space $\mathcal{D}^p_{p+\alpha}$ and the spaces H^p_α and S^p_α, introduced by Mateljević and Pavlović (1983), contain the same univalent functions whenever $2 \le p < \infty$ and $-2 < \alpha < \infty$. This, combined with an earlier result by Baernstein, Girela and Peláez (2004), yields Corollary 2 which states that $H^p \cap \mathcal{U} = \mathcal{D}^p_{p-1} \cap \mathcal{U} = S^p_{-1} \cap \mathcal{U}$ for all $0 < p < \infty$.

Pommerenke (1977) proved that univalent Bloch functions belong to $BMOA$, the space of analytic functions in the Hardy space H^1 with boundary values of bounded mean oscillation. Aulaskari, Lappan, Xiao and Zhao (1997) improved this result by showing that $\mathcal{B} \cap \mathcal{U} = Q_p \cap \mathcal{U}$ for any $0 < p \le 1$. Theorem 4 gives several new characterizations of univalent functions in the Bloch space $\mathcal{B}$. It shows that there exists in a sense a much smaller Möbius invariant subspace of the Bloch space than Q_p still containing all univalent Bloch functions. Therefore Theorem 4 improves the result by Aulaskari et. al.

The results of this paper are introduced in Section 2 after necessary definitions and a brief survey and several observations on the known results. In addition to the results mentioned above, univalent functions in the Besov type and the α-Bloch spaces are also studied. The results for the corresponding "little" spaces are also given. Sections 3-9 contain the proofs of the results in chronological order.

2. Background and results

Let $\mathcal{H}(\mathbb{D})$ denote the algebra of all analytic functions in the unit disc $\mathbb{D} := \{z : |z| < 1\}$. For $0 < p \le \infty$, the Hardy space H^p consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{H^p} := \lim_{r \to 1^-} M_p(r, f) < \infty,$$

where

$$M_p(r, f) := \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{\frac{1}{p}}, \quad 0 < p < \infty,$$

are the standard L^p-means of the restriction of f to the circle of radius r centered at the origin, and

$$M_\infty(r,f) := \max_{0\le\theta\le 2\pi} |f(re^{i\theta})|$$

is the maximum modulus function. For the theory of Hardy spaces, see [11, 16, 21].

A function $f \in \mathcal{H}(\mathbb{D})$ is said to be univalent if it is one-to-one. The class of all univalent functions in $\mathbb{D}$ is denoted by $\mathcal{U}$. For the theory of univalent functions, see [12, 29, 31].

The following characterization of univalent functions in Hardy spaces is due to Hardy and Littlewood [20], Pommerenke [27] and Prawitz [32].

Theorem A. *Let $0 < p < \infty$ and $f \in \mathcal{U}$. Then $f \in H^p$ if and only if*

$$\int_0^1 M_\infty^p(r,f)\,dr < \infty.$$

Moreover, if $0 < p < 2$, then $f \in H^p$ if and only if

$$\int_0^1 M_1^p(r,f')\,dr < \infty.$$

In fact, $\int_0^1 M_\infty^p(r,f)\,dr \le \pi\|f\|_{H^p}^p$ for all $f \in \mathcal{H}(\mathbb{D})$, see [7, p. 841] or [27, Hilfssatz 1] for a proof. The opposite implication follows by Prawitz's [32] result which states that $M_p^p(r,f) \le p\int_0^r M_\infty(\rho,f)^p\rho^{-1}\,d\rho$ for all $f \in \mathcal{U}$ such that $f(0) = 0$, see [29, p. 127]. A proof of the second part of Theorem A can be found in [27].

For $0 < p < \infty$ and $-1 < \alpha < \infty$, the weighted Bergman space A^p_α consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{A^p_\alpha}^p := \int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^\alpha\,dA(z) < \infty,$$

where dA denotes the element of the Lebesgue area measure on $\mathbb{D}$. For the theory of Bergman spaces, see [13, 19].

Univalent functions in weighted Bergman spaces have been recently characterized by Baernstein, Girela and Peláez [7] and Peláez [25].

Theorem B. *Let $0 < p < \infty$, $-1 < \alpha < \infty$ and $f \in \mathcal{U}$. Then $f \in A^p_\alpha$ if and only if*

$$\int_0^1 r(1-r^2)^\alpha\left(\int_0^r M_\infty^p(\rho,f)\,d\rho\right)dr < \infty.$$

Moreover, if $0 < p < 2$, then $f \in A^p_\alpha$ if and only if

$$\int_0^1 r(1-r^2)^\alpha\left(\int_0^r M_1^p(\rho,f')\,d\rho\right)dr < \infty.$$

As usual, the Hardy space H^p is identified with the limit space of the weighted Bergman space A^p_α as $\alpha \to -1^+$, and therefore the notation $A^p_{-1} := H^p$ is adopted. One reason to do so is that $\lim_{\alpha\to-1^+} \|f\|_{A^p_\alpha} = \|f\|_{H^p}$, see [46]. On the other hand, a generalization of the Littlewood-Paley formula due to Stein states that

$$\|f\|^p_{H^p} = \frac{p^2}{2}\int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2 \log\frac{1}{|z|}\,dA(z) + |f(0)|^p\,, \tag{2.1}$$

see [37], and also [36, 42]. An analogous formula for the weighted Bergman space exists, namely

$$\|f\|^p_{A^p_\alpha} \simeq \int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2 \left(\log\frac{1}{|z|}\right)^{\alpha+2} dA(z) + |f(0)|^p\,, \tag{2.2}$$

see, for example, [35]. (Here and from now on the symbol $\simeq$ means that the quantities in other sides of the symbol are comparable, that is, their quotient is bounded and bounded away from zero. Moreover, the notation $a \lesssim b$ means that $a \le Cb$ for some positive constant C, independent of a and b, and $a \gtrsim b$ is understood in an analogous manner.) In view of these facts it is natural to expect that Theorem A is the limit case of Theorem B as $\alpha \to -1^+$. Of course, $\alpha = -1$ can not be substituted in the statement of Theorem B since the singularities would become too strong. However, an application of Fubini's theorem yields

$$\begin{aligned}\int_0^1 r(1-r^2)^\alpha \left(\int_0^r M^p_\infty(\rho,f)\,d\rho\right) dr &= \int_0^1 M^p_\infty(\rho,f)\int_\rho^1 r(1-r^2)^\alpha\,dr\,d\rho\\ &= \frac{1}{2(\alpha+1)}\int_0^1 M^p_\infty(\rho,f)(1-\rho^2)^{\alpha+1}\,d\rho,\end{aligned}$$

and similarly

$$\int_0^1 r(1-r^2)^\alpha \left(\int_0^r M^p_1(\rho,f')\,d\rho\right) dr = \frac{1}{2(\alpha+1)}\int_0^1 M^p_1(\rho,f')(1-\rho^2)^{\alpha+1}\,d\rho.$$

This shows that Theorem B indeed generalizes Theorem A for the weighted Bergman spaces, and thus the following result holds.

Theorem C. *Let $0 < p < \infty$, $-1 \le \alpha < \infty$ and $f \in \mathcal{U}$. Then $f \in A^p_\alpha$ if and only if*

$$J^p_\alpha(f) := \int_0^1 M^p_\infty(r,f)(1-r^2)^{\alpha+1}\,dr < \infty.$$

Moreover, if $0 < p < 2$, then $f \in A^p_\alpha$ if and only if

$$K^p_\alpha(f) := \int_0^1 M^p_1(r,f')(1-r^2)^{\alpha+1}\,dr < \infty.$$

The second part of the assertion in Theorem C is true in the case $p = 1$ and $-1 < \alpha < \infty$ for all $f \in \mathcal{H}(\mathbb{D})$. To see this, it suffices to observe that $\|f\|_{A^1_\alpha} \simeq \|f'\|_{A^1_{\alpha+1}} + |f(0)| \simeq K^1_\alpha(f) + |f(0)|$, where the first asymptotic equality follows by the well-known result $\|f\|_{A^p_\alpha} \simeq \|f'\|_{A^p_{p+\alpha}} + |f(0)|$ for all $0 < p < \infty$ and $-1 < \alpha < \infty$, and the second one is a simple consequence of the fact that $M^p_1(r, f')$ is an increasing function of r.

Let $\Delta(0, r) := \{z : |z| < r\}$. For $0 < p < \infty$ and $-2 < \alpha < \infty$, the spaces H^p_α and S^p_α consist of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|^p_{H^p_\alpha} := \int_0^1 r(1-r^2)^{\alpha+1} \left(\int_{\Delta(0,r)} |f'(z)|^2 |f(z)|^{p-2} \, dA(z) \right) dr < \infty$$

and

$$\|f\|^p_{S^p_\alpha} := \int_0^1 r(1-r^2)^{\alpha+1} \left(\int_{\Delta(0,r)} |f'(z)|^2 \, dA(z) \right)^{\frac{p}{2}} dr < \infty,$$

respectively. These spaces were introduced and studied by Mateljević and Pavlović [24]. Obviously, $H^2_\alpha = S^2_\alpha$ for all $-2 < \alpha < \infty$. Moreover, Fubini's theorem shows that

$$\|f\|^p_{H^p_\alpha} = \frac{1}{2(\alpha+2)} \int_{\mathbb{D}} |f'(z)|^2 |f(z)|^{p-2} (1-|z|^2)^{\alpha+2} \, dA(z), \tag{2.3}$$

and therefore $H^p_\alpha = A^p_\alpha$ for all $\alpha \geq -1$ by (2.1) and (2.2). In particular, $H^p_{-1} = A^p_{-1} = H^p$ for all $0 < p < \infty$. The definition of S^p_α is in a sense of geometric nature since $\int_{\Delta(0,r)} |f'(z)|^2 \, dA(z)$ is the area of image of $\Delta(0, r)$ under f counting multiplicities. It is known that the spaces H^p_α and S^p_α obey the strict inclusions

$$H^p_\alpha \subsetneq S^p_\alpha, \quad 0 < p < 2,$$

and

$$S^p_\alpha \subsetneq H^p_\alpha, \quad 2 < p < \infty, \tag{2.4}$$

see [24]. Girela, Pavlović and Peláez proved the following result in [17].

Theorem D. *Let $0 < p \leq 2$, $-2 < \alpha < \infty$ and $f \in \mathcal{U}$. Then the following assertions are equivalent:*

(1) $f \in H^p_\alpha$;

(2) $f \in S^p_\alpha$;

(3) $J^p_\alpha(f) < \infty$.

For $0 < p < \infty$ and $-1 < \beta < \infty$, the Dirichlet type space $\mathcal{D}^p_\beta$ consists of those $f \in \mathcal{H}(\mathbb{D})$ for which $\|f\|_{\mathcal{D}^p_\beta} := \|f'\|_{A^p_\beta} < \infty$, and the true norm in $\mathcal{D}^p_\beta$ is defined by $\|f\|_{\mathcal{D}^p_\beta} + |f(0)|$.

It is natural to ask how the Dirichlet type space $\mathcal{D}^p_{p+\alpha}$ is related to the spaces H^p_α and S^p_α. An immediate observation is that (2.1) and (2.3) yield $\mathcal{D}^2_1 = H^2 = H^2_{-1}$. Further, it is known that

$$\mathcal{D}^p_{p-1} \subset H^p, \quad 0 < p \leq 2, \tag{2.5}$$

and

$$H^p \subset \mathcal{D}^p_{p-1}, \quad 2 \leq p < \infty, \tag{2.6}$$

and the inclusions are strict when $p \neq 2$. The inclusion (2.5) for $1 \leq p \leq 2$ can be proved by Riesz-Thorin interpolation theorem, and the case $0 < p < 1$ has been proved by Flett [15], an alternative proof by Vinogradov can be found in [38]. The inclusion (2.6) follows by a classical result due to Littlewood and Paley [22], see also the proof by Luecking [23]. An easy way to see that the inclusions (2.5) and (2.6) are strict is to use functions with Hadamard gaps (lacunary series). It is well known that if the power series representation $\sum_{k=0}^\infty a_k z^{n_k}$ of f satisfies $n_{k+1}/n_k > \lambda > 1$ for all k, then

$$f \in H^p \Leftrightarrow \sum_{k=0}^\infty |a_k|^2 < \infty, \qquad f \in \mathcal{D}^p_\beta \Leftrightarrow \sum_{k=0}^\infty |a_k|^p n_k^{p-\beta-1} < \infty, \tag{2.7}$$

see, for example, [47, Chapter V in Vol. I] and [43, Theorem 5.5], respectively. It is worth noticing that the proof of the inclusion (2.6) by Luecking can be modified such that, in the setting of the space H^p_α, it covers other values of α than just $\alpha = -1$ which corresponds to the Hardy space H^p. In the case of real valued harmonic functions it yields the following result which deserves to be stated separately.

Theorem E. *Let $2 \leq p < \infty$, $-\infty < \alpha < \infty$ such that $p+\alpha > -1$, and let u be a real valued harmonic function on $\overline{\mathbb{D}}$. Then there exists a positive constant C, depending only on p and α, such that*

$$\int_{\mathbb{D}} |\nabla u(z)|^p (1-|z|^2)^{p+\alpha}\, dA(z) \leq C \int_{\mathbb{D}} |u(z)|^{p-2} |\nabla u(z)|^2 \left(1-|z|^2\right)^{\alpha+2} dA(z).$$

The proof is based on the local estimate

$$|\nabla u(0)|^p \leq 2^{2p-1} p(p-1) \int_{\Delta(0,\frac{1}{2})} |u(z)|^{p-2} |\nabla u(z)|^2 \log \frac{1}{2|z|}\, dA(z),$$

see (2.4) in [23], which gives

$$|\nabla u(w)|^p(1-|w|^2)^p \lesssim \int_{\Delta(0,\frac{1}{2})} |(u\circ\varphi_w)(z)|^{p-2}|(\nabla u)\circ\varphi_w(z)|^2|\varphi_w'(z)|^2 \log\frac{2}{|z|}\,dA(z)$$

when applied to the function $u\circ\varphi_w$, where $\varphi_w(z) := (w-z)/(1-\bar{w}z)$ is the automorphism of $\mathbb{D}$ which interchanges the origin and the point $w\in\mathbb{D}$. Now by integrating this inequality with respect to $(1-|w|^2)^\alpha\,dA(w)$ and following the original proof, the assertion follows. The only extra fact needed is the asymptotic equality $(1-|w|^2)\simeq(1-|\varphi_w(z)|^2)$ which holds when z belongs to a compact subset of $\mathbb{D}$.

A detailed proof of the inclusion $H^p_\alpha\subset\mathcal{D}^p_{p+\alpha}$, $2\le p<\infty$, is presented in Section 3.3. This combined with (2.4) gives

$$S^p_\alpha\subset H^p_\alpha\subset\mathcal{D}^p_{p+\alpha},\quad 2\le p<\infty.$$

It remains open wether or not $\mathcal{D}^p_{p+\alpha}$ is contained in H^p_α for $0<p<2$ and $-2<\alpha<-1$ such that $p+\alpha>-1$. However, if $f\in\mathcal{U}$ and $0<p\le 1$, then (5.2), Section 4.2 and Lemma I in Section 3.1 yield

$$\|f\|^p_{H^p_\alpha}\lesssim J^p_\alpha(f)\lesssim K^p_\alpha(f)\lesssim\|f\|^p_{\mathcal{D}^p_{p+\alpha}},$$

and thus $\mathcal{D}^p_{p+\alpha}\cap\mathcal{U}\subset H^p_\alpha\cap\mathcal{U}$ for $0<p\le 1$.

Baernstein, Girela and Peláez proved the following result in [7]. It shows that even if H^p and $\mathcal{D}^p_{p-1}$ are different unless $p=2$, univalent functions in H^p are precisely those in $\mathcal{D}^p_{p-1}$.

Theorem F. *Let $0<p<\infty$ and $f\in\mathcal{U}$. Then $f\in H^p$ if and only if $f\in\mathcal{D}^p_{p-1}$, and moreover,*

$$\|f\|^p_{H^p}\simeq\|f\|^p_{\mathcal{D}^p_{p-1}}+|f(0)|^p\simeq J^p_{-1}(f),\quad 0<p<\infty.$$

Baernstein, Girela and Peláez pointed out that Theorem F yields

$$\mathcal{U}\subset\bigcap_{0<p<\frac{1}{2}}\mathcal{D}^p_{p-1},\tag{2.8}$$

which improves the known fact $\mathcal{U}\subset\cap_{0<p<\frac{1}{2}}H^p$ by (2.5). However, similar applications as in [7, pp. 847–848] of results due to Feng and MacGregor [14] on the growth of the integral means $M_p(r,f')$ of univalent functions show that

$$\mathcal{U}\subset\bigcap_{c(\alpha)<p<1+\frac{\alpha}{2}}\mathcal{D}^p_{p+\alpha},$$

where $c(\alpha) := \max\{-(\alpha+1), -2(\alpha+1)\}$. This intersection is clearly smaller than the one in (2.8). For example, the function $\sum_{k=1}^{\infty} 2^{-k\varepsilon} z^{2^k}$, $\varepsilon > 0$, belongs to $\mathcal{D}^p_{p-1}$ for all $p > 0$ since $\sum_{k=1}^{\infty} 2^{-pk\varepsilon}$ converges, but it does not belong to $\mathcal{D}^p_{p+\alpha}$ if $\alpha < -1$ and $\varepsilon < -\frac{\alpha+1}{p}$ since $\sum_{k=1}^{\infty} 2^{-pk\varepsilon-k(\alpha+1)}$ clearly diverges, see (2.7).

The following result shows that univalent functions in $\mathcal{D}^p_{p+\alpha}$, H^p_α and S^p_α are the same when $2 \le p < \infty$. Since $H^p = H^p_{-1}$ and $A^p_\alpha = H^p_\alpha = \mathcal{D}^p_{p+\alpha}$ for $-1 < \alpha < \infty$, it completes in part Theorems C, D and F, and thus Theorems A and B also.

Theorem 1. *Let $2 \le p < \infty$ and $-2 < \alpha < \infty$, and let $f \in \mathcal{U}$. Then the following conditions are equivalent:*

(1) $f \in \mathcal{D}^p_{p+\alpha}$;

(2) $f \in H^p_\alpha$;

(3) $f \in S^p_\alpha$;

(4) $J^p_\alpha(f) < \infty$.

Theorem 1 combined with Theorems D and F yields the following result.

Corollary 2. *Let $0 < p < \infty$ and $-2 < \alpha < \infty$. Then*

$$H^p_\alpha \cap \mathcal{U} = S^p_\alpha \cap \mathcal{U},$$

and, in particular,

$$H^p \cap \mathcal{U} = S^p_{-1} \cap \mathcal{U} = \mathcal{D}^p_{p-1} \cap \mathcal{U}.$$

Corollary 2 and the geometric interpretation of space S^p_α implies that $f \in \mathcal{U}$ belongs to the weighted Bergman space $A^p_\alpha = H^p_\alpha$, $-1 \le \alpha < \infty$, if and only if

$$\int_0^1 (\text{Area} f(\Delta(0,r)))^{\frac{p}{2}} (1-r^2)^{\alpha+1}\, dr < \infty, \tag{2.9}$$

in particular $f \in H^p$ if and only if (2.9) with $\alpha = -1$ is satisfied. Theorem C and (2.9) show that if $f \in \mathcal{U}$, then $M_\infty(r,f)$ and $\sqrt{\text{Area} f(\Delta(0,r))}$ are of the same growth measured in terms of $L^p((1-r)^{\alpha+1}dr)$. For different results involving geometric conditions and related to Hardy and weighted Bergman spaces, see [7, 25].

In view of Theorem 1 and Corollary 2 it is natural to expect that the sets of univalent functions in $\mathcal{D}^p_{p+\alpha}$, H^p_α and S^p_α would coincide also when $0 < p < 2$ and $-2 < \alpha < -1$ such that $p + \alpha > -1$. This, however, turns out to be false. In order to see this, a couple of observations on the quantities $J^p_\alpha(f)$ and

$K^p_\alpha(f)$ are made. The asymptotic inequality $J^p_\alpha(f) \lesssim K^p_\alpha(f)$ follows by the fact $M_\infty(r,f) \le \pi r M_1(r,f') + |f(0)|$ which holds for all $f \in \mathcal{U}$ by Section 4.2. On the other hand, by Theorem C, the quantities $J^p_\alpha(f)$ and $K^p_\alpha(f)$ are comparable if $0 < p < 2$ and $\alpha \ge -1$. In view of $J^p_\alpha(f) \lesssim K^p_\alpha(f)$, this also follows directly by using the (asymptotic) inequalities

$$\int_0^r M_1^p(s,f')\,ds \lesssim M_p^p(r,f) \le p\int_0^r M_\infty^p(s,f)\frac{ds}{s},$$

see [27, Theorem 4] and [29, p. 127]. However, the finiteness of $J^p_\alpha(f)$ does not necessarily imply the finiteness of $K^p_\alpha(f)$ in the case $-2 < \alpha < -1$. To see this, it suffices to notice that Pommerenke has constructed a bounded univalent function f with non-negative Taylor coefficients a_n such that $a_n \ne O(n^{-0.84})$, see [28]. This implies $M_1(r,f') \ne O((1-r)^{-0.16})$, and therefore $K^p_\alpha(f)$ does not remain finite for a given $0 < p < 2$ if α is chosen to be sufficiently close to -2, yet $J^p_\alpha(f)$ is clearly finite for any $-2 < \alpha < \infty$. Since $K^1_\alpha(f) \simeq \|f\|_{\mathcal{D}^1_{1+\alpha}}$ for $f \in \mathcal{H}(\mathbb{D})$, this also shows that

$$H^\infty \cap \mathcal{U} \not\subset \mathcal{D}^1_{1+\alpha}, \quad -2 < \alpha \le -1.84. \tag{2.10}$$

For results on coefficient problems of univalent functions, see [6, 12, 18, 29].

The following result shows that univalent functions in $\mathcal{D}^p_{p+\alpha}$ can be characterized by the condition $J^p_\alpha(f) < \infty$ in certain special cases.

Theorem 3. *Let $0 < p < 1$ and $-\frac{475}{316} < \alpha < -1$ such that $p > -\frac{316}{159}(\alpha+1)$, and let $f \in \mathcal{U}$. Then the following conditions are equivalent:*

(1) $f \in \mathcal{D}^p_{p+\alpha}$;

(2) $J^p_\alpha(f) < \infty$;

(3) $K^p_\alpha(f) < \infty$.

The mysterious fractional numbers $\frac{475}{316}$ and $\frac{316}{159}$ in the statement come from Baernstein's [6] result related to coefficient problems.

It remains open wether or not the condition $K^p_\alpha(f) < \infty$ characterizes univalent functions in $\mathcal{D}^p_{p+\alpha}$ when $0 < p < 2$. Clearly, $\|f\|_{\mathcal{D}^1_{1+\alpha}} \simeq K^1_\alpha(f)$, and moreover Lemma I implies that one of the conditions $f \in \mathcal{D}^p_{p+\alpha}$ and $K^p_\alpha(f) < \infty$ always implies the other.

The space $\mathcal{D}^p_{p-2}$ is the classical Besov space B^p. Univalent functions in Besov spaces have been characterized by Walsh [39], see also the related results by Donaire, Girela and Vukotić [10]. The following result due to Walsh is of geometric nature.

Theorem G. *Let $1 < p < \infty$ and let Ω be a simply connected proper subdomain of the complex plane. Let $f \in \mathcal{U}$ and $f(\mathbb{D}) = \Omega$. Then $f \in B^p$ if and only if $\int_\Omega d_\Omega(w)^{p-2}\, dA(w) < \infty$, where $d_\Omega(w)$ stands for the Euclidean distance from w to the boundary of Ω.*

We turn now to consider univalent Bloch functions. The Bloch space $\mathcal{B}$ consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}} := \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2) < \infty.$$

For the theory of Bloch spaces, see the classical reference [1], and also [31, 45].

If Ω is a simply connected proper subdomain of the complex plane and $f \in \mathcal{U}$ such that $f(\mathbb{D}) = \Omega$, then

$$d_\Omega(f(z)) \le |f'(z)|(1-|z|^2) \le 4 d_\Omega(f(z)) \tag{2.11}$$

for all $z \in \mathbb{D}$, see, for example, [31]. It follows that $f \in \mathcal{B}$ if and only if $\sup_{w\in\Omega} d_\Omega(w) < \infty$. In other words, $f \in \mathcal{B}$ if and only if the image of $\mathbb{D}$ under f does not contain arbitrarily large discs. This should be compared with Theorem G.

Let the Green's function of $\mathbb{D}$ with a logarithmic singularity at $a \in \mathbb{D}$ be defined by $g(z,a) := -\log|\varphi_a(z)|$, where $\varphi_a(z) = (a-z)/(1-\bar{a}z)$ is the automorphism of $\mathbb{D}$ which interchanges the origin and the point a. Recall that φ_a is its own inverse and satisfies the well-known equalities

$$1 - |\varphi_a(z)|^2 = |\varphi_a'(z)|(1-|z|^2) = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}.$$

For $0 < p < \infty$ and $0 < s < \infty$ such that $p + s > 1$, the Möbius invariant Besov type space B^p_s consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|^p_{B^p_s} := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} g^s(z,a)\, dA(z) < \infty.$$

The space B^2_s is the Q_s-space, in particular, $B^2_1 = BMOA$, the space of analytic functions in the Hardy space H^1 whose boundary values have bounded mean oscillation on the unit circle. For the theory of Q_s-spaces, see [40, 41]. For $1 < s < \infty$, the Besov type space B^p_s coincides with the Bloch space and $\|f\|_{\mathcal{B}} \simeq \|f\|_{B^p_s}$, see [44]. It is also well know that

$$\|f\|^p_{B^p_s} \simeq \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} (1-|\varphi_a(z)|^2)^s\, dA(z),$$

and therefore the change of variable $w = \varphi_a(z)$ yields

$$\|f\|_{B^p_s} \simeq \sup_{a\in\mathbb{D}} \|f\circ\varphi_a\|_{\mathcal{D}^p_{p-2+s}}. \tag{2.12}$$

Theorem H, due to Aulaskari, Lappan, Xiao and Zhao [2] and Pommerenke [30], shows that univalent Bloch functions belong to Q_s for all $s > 0$.

Theorem H. *Let $0 < s \le 1$ and $f \in \mathcal{U}$. Then $f \in \mathcal{B}$ if and only if $f \in Q_s$.*

It is worth noticing that Theorem F implies $BMOA \cap \mathcal{U} = \cap_{p>0} B^p_1 \cap \mathcal{U}$. This is due to the facts $\|f\|_{BMOA} \simeq \sup_{a\in\mathbb{D}} \|f\circ\varphi_a - f(a)\|_{H^p}$ and $\|f\|_{B^p_1} \simeq \sup_{a\in\mathbb{D}} \|f\circ\varphi_a\|_{\mathcal{D}^p_{p-1}}$, see [5] and (2.12), respectively. However, this observation does not yield an improvement of Theorem H since the set $\cap_{p>0} B^p_1$ is in a sense much larger than $\cap_{s>0} Q_s$. More precisely, it is known that $\cup_{0<s<1} Q_s \subsetneq \cap_{p>0} B^p_1$, see [34, Section 2.2].

The following result gives several equivalent characterizations of univalent functions in the Bloch space $\mathcal{B}$.

Theorem 4. *Let $0 < p < \infty$ and $-2 < \alpha < \infty$, and let $f \in \mathcal{U}$. Then the following assertions are equivalent:*

(1) $f \in \mathcal{B}$;

(2) $\sup_{a\in\mathbb{D}} \|f\circ\varphi_a - f(a)\|_{H^p_\alpha} < \infty$;

(3) $\sup_{a\in\mathbb{D}} \|f\circ\varphi_a\|_{S^p_\alpha} < \infty$;

(4) $\sup_{a\in\mathbb{D}} J^p_\alpha(f\circ\varphi_a - f(a)) < \infty$.

The statement of Theorem 4 should be interpreted as follows. If f is a univalent Bloch function then the conditions (2) and (3) are satisfied for any $0 < p < \infty$ and any $-2 < \alpha < \infty$, no matter how large or small they are. Converse is true for all $f \in \mathcal{H}(\mathbb{D})$; if f satisfies one of the conditions (2) and (3) for some $0 < p < \infty$ and $-2 < \alpha < \infty$, then f is a Bloch function. To see this for the condition (2), it suffices to observe that $H^p_\alpha = A^p_\alpha$ for $\alpha > -1$, and $f \in \mathcal{B}$ if and only if $\sup_{a\in\mathbb{D}} \|f\circ\varphi_a - f(a)\|_{A^p_\alpha} < \infty$. For the proof of the fact (3) implies $f \in \mathcal{B}$, see Section 5.4. It is not a surprise that the conditions (2) and (3) imply $f \in \mathcal{B}$. In fact, Rubel and Timoney [33] have shown that if X is a"reasonable" subspace of $\mathcal{H}(\mathbb{D})$ such that the map $f \to f\circ\varphi_a$ is uniformly bounded on X, then X must be a subset of the Bloch space. Moreover, the condition (4) is always satisfied if $f \in \mathcal{B}$, see Section 5.1.

Theorem 4 contains Theorem H as a special case since $f \in Q_{2+\alpha}$ if and only if the condition (3) of Theorem 4 for $p = 2$ is satisfied. Moreover, the following immediate consequence of Theorem 4 improves Theorem H.

Corollary 5. *Denote by $\bar{S}^p_\alpha$ the set of those $f \in \mathcal{H}(\mathbb{D})$ for which the condition (3) of Theorem 4 is satisfied. Then*

$$\mathcal{B} \cap \mathcal{U} = \bigcap_{p>0,\alpha>-2} \bar{S}^p_\alpha \cap \mathcal{U}.$$

By Hölder's inequality, $\|f \circ \varphi_a\|_{S^q_\alpha} \lesssim \|f \circ \varphi_a\|_{S^p_\alpha}$ for $0 < q < p < \infty$, and therefore $\bar{S}^p_\alpha \subset Q_{2+\alpha}$ for all $2 < p < \infty$. Moreover, this inclusion is strict since the function $f(z) = \sum_{k=1}^{\infty} 2^{k(\frac{\alpha+1}{2}-\varepsilon)} z^{2^k}$, where $\varepsilon := 1 + \alpha(\frac{1}{2} - \frac{1}{p})$ and $p > 2$, belongs to $Q_{2+\alpha}$ for all $\alpha > -2$ but does not belong to S^p_α, and thus does not satisfy (3), see [4] and [24], respectively.

If $g \in \mathcal{U}$ is zero-free and $g(\mathbb{D}) = \Omega$, then $f := \log g \in \mathcal{H}(\mathbb{D})$ and (2.11) yields

$$\frac{d_\Omega(g(z))}{|g(z)|} \le \frac{|g'(z)|(1-|z|^2)}{|g(z)|} = |f'(z)|(1-|z|^2) \le 4\frac{d_\Omega(g(z))}{|g(z)|}. \tag{2.13}$$

Since $d_\Omega(g(z)) \le |g(z)|$ it follows that $f \in \mathcal{B}$, and therefore Theorem 4 yields the following result.

Corollary 6. *If $f \in \mathcal{U}$ is zero-free, then*

$$\log f \in \bigcap_{p>0,\alpha>-2} \bar{S}^p_\alpha.$$

The next result is a consequence of Theorem 3 and it should be compared with Theorem H. It shows that at least for some values of $0 < p < 2$ and $-2 < \alpha < -1$, univalent Bloch functions belong to the Besov type space $B^p_{2+\alpha}$. It seems difficult to determine all the values of p and α for which $\mathcal{B} \cap \mathcal{U} = B^p_{2+\alpha} \cap \mathcal{U}$.

Theorem 7. *Let $0 < p < 1$ and $-\frac{475}{316} < \alpha < -1$ such that $p > -\frac{316}{159}(\alpha+1)$, and let $f \in \mathcal{U}$. Then $f \in \mathcal{B}$ if and only if $f \in B^p_{2+\alpha}$.*

The following consequence of Theorem 7 follows by the fact $B^p_s \subset B^1_s$ for $0 < p < 1$. Note that $\frac{157}{316} \approx 0.497 < \frac{1}{2}$.

Corollary 8. *Denote $s_0 := \frac{157}{316}$. Then*

$$\mathcal{B} \cap \mathcal{U} = \bigcap_{s>s_0} B^1_s \cap \mathcal{U}.$$

Since $H^\infty \subset \mathcal{B}$ and $B^1_s \subset \mathcal{D}^1_{s-1}$, the assertion in Corollary 8 fails for $s_0 < 0.16$ by (2.10).

We now proceed to consider univalent functions in the little Bloch space. The little Bloch space $\mathcal{B}_0$ is the closure of polynomials in the Bloch space and consists of those $f \in \mathcal{H}(\mathbb{D})$ for which $|f'(z)|(1-|z|^2) \to 0$ as $|z| \to 1^-$. By (2.11), $f \in \mathcal{U}$ belongs to $\mathcal{B}_0$ if and only if $\lim_{|z|\to 1} d_\Omega(f(z)) = 0$. The following result is the counterpart of Theorem 4 for the little Bloch space.

Theorem 9. *Let $0 < p < \infty$ and $-2 < \alpha < \infty$, and let $f \in \mathcal{U}$. Then the following assertions are equivalent:*

(1) $f \in \mathcal{B}_0$;

(2) $\displaystyle\lim_{|a|\to 1^-} \|f\circ\varphi_a - f(a)\|_{H^p_\alpha} = 0$;

(3) $\displaystyle\lim_{|a|\to 1^-} \|f\circ\varphi_a\|_{S^p_\alpha} = 0$;

(4) $\displaystyle\lim_{|a|\to 1^-} J^p_\alpha(f\circ\varphi_a - f(a)) = 0$.

The condition (3) of Theorem 9 for $p = 2$ is satisfied if and only if $f \in Q_{2+\alpha,0}$, and therefore Theorem 9 improves the known fact $\mathcal{B}_0 \cap \mathcal{U} = Q_{s,0} \cap \mathcal{U}$ for all $0 < s \le 1$, see [2, 30]. Denote by $\bar{S}^p_{\alpha,0}$ the set of those $f \in \mathcal{H}(\mathbb{D})$ for which the condition (3) of Theorem 9 is satisfied. Then it follows that

$$\mathcal{B}_0 \cap \mathcal{U} = \bigcap_{p>0,\alpha>-2} \bar{S}^p_{\alpha,0} \cap \mathcal{U}.$$

Moreover, the "little oh" counterpart of Corollary 6 states that if $g \in \mathcal{U}$ is zero-free and $g(\mathbb{D}) = \Omega$, then

$$\log g \in \bigcap_{p>0,\alpha>-2} \bar{S}^p_{\alpha,0} \Leftrightarrow \frac{d_\Omega(w)}{|w|} \to 0, \quad |w| \to 0, \infty,$$

see (2.13).

The little Besov type space $B^p_{s,0}$ consists of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\lim_{|a|\to 1^-} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} g^s(z,a)\, dA(z) = 0.$$

It is known that $B^p_{s,0}$ is a subspace of the little Bloch space and $B^p_{s,0} = \mathcal{B}_0$ for $s > 1$, see [44]. The following result is an immediate consequence of Theorem 3.

Theorem 10. *Let $0 < p < 1$ and $-\frac{475}{316} < \alpha < -1$ such that $p > -\frac{316}{159}(\alpha+1)$, and let $f \in \mathcal{U}$. Then $f \in \mathcal{B}_0$ if and only if $f \in B^p_{2+\alpha,0}$.*

It is worth pointing out that the nesting property of the spaces $B^p_{s,0}$ with respect to p yields the following counterpart of Corollary 8:

$$\mathcal{B}_0 \cap \mathcal{U} = \bigcap_{s>s_0} B^1_{s,0}, \quad s_0 = \frac{157}{316}.$$

Finally, a couple of observations on α-Bloch spaces are made. The α-Bloch space $\mathcal{B}^\alpha$ consists of those $f \in \mathcal{H}(\mathbb{D})$ for which $|f'(z)|(1-|z|^2)^\alpha$ is uniformly bounded in $\mathbb{D}$. Since $f \in \mathcal{B}^\alpha$, $1 < \alpha < \infty$, if and only if $M_\infty(r,f) = O((1-r)^{1-\alpha})$ as $r \to 1^-$, it follows that $\mathcal{U} \subset \mathcal{B}^3$. Denote by $B^p_{\alpha,s}$ the set of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|^p_{B^p_{\alpha,s}} := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{\alpha p-2} g^s(z,a)\, dA(z) < \infty.$$

The space $B^p_{\alpha,s}$ is a subspace of $\mathcal{B}^\alpha$, and $B^p_{\alpha,s} = \mathcal{B}^\alpha$ for $s > 1$, see [44]. The following result shows that univalent α-Bloch functions belong to $B^p_{\alpha,s}$ under certain conditions on the parameters.

Theorem 11. *If $\frac{3}{2} - \frac{1}{320} < \alpha \le 3$, $1 \le p < \infty$ and $0 < s \le 1$. Then the following assertions are equivalent:*

(1) $f \in \mathcal{B}^\alpha$;

(2) $M_1(r,f') = O\left(\left(\frac{1}{1-r}\right)^{\alpha-1}\right), \quad r \to 1^-$;

(3) $f \in B^p_{\alpha,s}$.

The deep content of this result is the implication (1) ⇒ (2) which is in fact [6, Theorem 1]. It is clear that an analogous result for little α-Bloch space $\mathcal{B}^\alpha_0$ (the closure of polynomials in $\mathcal{B}^\alpha$) can be established. In order to avoid unnecessary repetition, the details are omitted.

3. Proof of Theorem 1

It will be shown that (1) ⇒ (4) ⇒ (3) ⇒ (2) ⇒ (1). Since (3) ⇒ (2) by (2.4), it remains to prove the other three implications. Note that the only implication which uses the property $f \in \mathcal{U}$ is (4) ⇒ (3).

3.1. Proof of $(1) \Rightarrow (4)$

Two different proofs for this implication will be given. The first one follows the reasoning in [7, p. 849]. In order to do so the following variant of [11, Theorem 5.9], to be found in [7, p. 841], is needed.

Lemma I. *Let* $0 < p < q \leq \infty$ *and* $f \in \mathcal{H}(\mathbb{D})$*. There exists a positive constant* C*, depending only on* p *and* q*, such that*

$$M_q(r,f) \leq C M_p\left(\frac{1+r}{2}, f\right)(1-r)^{\frac{1}{q}-\frac{1}{p}}, \quad 0 \leq r < 1.$$

The first proof consists of three steps. First, Lemma I is applied in order to estimate $M_\infty(r,f)$ in terms of $M_{\beta p}(r,f)$, where $\beta > 1$ is a suitably chosen (possibly large) constant. The second step is an application of a Hardy-Littlewood inequality which relates the growth of the integral means of $f \in \mathcal{H}(\mathbb{D})$ with those of its derivative. As the third step, Lemma I is applied again in order to estimate $M_{\beta p}(r,f')$ in terms of $M_p(r,f')$. The proof goes as follows. Let $0 < p < \infty$ and $-2 < \alpha < \infty$ such that $p + \alpha > -1$. Choose $\beta > 1$ sufficiently large such that $\alpha + 2 - \beta^{-1} > 0$. By Lemma I,

$$M_\infty^p(r,f) \lesssim \frac{M_{\beta p}^p(\frac{1+r}{2}, f)}{(1-\frac{1+r}{2})^{\frac{1}{\beta}}},$$

and it follows that

$$J_\alpha^p(f) \lesssim \int_0^1 M_{\beta p}^p(\rho, f)(1-\rho)^{\alpha+1-\beta^{-1}}\, d\rho.$$

By [15, Theorems 6 and 7] (see also [11, Theorem 5.6]),

$$\int_0^1 M_{\beta p}^p(\rho, f)(1-\rho)^{\alpha+1-\beta^{-1}}\, d\rho \lesssim \int_0^1 M_{\beta p}^p(\rho, f')(1-\rho)^{\alpha+1-\beta^{-1}+p}\, d\rho + |f(0)|^p,$$

and since Lemma I yields

$$M_{\beta p}^p(\rho, f') \lesssim \frac{M_p^p(\frac{1+\rho}{2}, f')}{(1-\frac{1+\rho}{2})^{1-\frac{1}{\beta}}},$$

it follows that

$$\begin{aligned} J_\alpha^p(f) &\lesssim \int_0^1 M_p^p(r, f')(1-r)^{\alpha+p}\, dr + |f(0)|^p \\ &\simeq \|f\|_{\mathcal{D}_{p+\alpha}^p}^p + |f(0)|^p. \end{aligned} \tag{3.1}$$

An alternative proof can be constructed as follows. By [15, Theorems 6 and 7] and Fubini's theorem,

$$
\begin{aligned}
J_\alpha^p(f) &\lesssim \int_0^1 M_\infty^p(r,f')(1-r)^{p+\alpha+1}\,dr + |f(0)|^p \\
&\simeq \int_0^1 (1-r^2)^{p+\alpha}\left(\int_0^r M_\infty^p(\rho,f')\,d\rho\right)dr + |f(0)|^p,
\end{aligned}
$$

and since $\int_0^r M_\infty^p(s,f)\,ds \le \pi M_p^p(r,f)$ for all $f\in\mathcal{H}(\mathbb{D})$, the assertion follows.

3.2. Proof of (4) ⇒ (3)

Let $0<p<\infty$ and $-2<\alpha<\infty$, and let $f\in\mathcal{U}$. Then

$$
\begin{aligned}
\frac{p}{2\pi}\int_{\Delta(0,r)} |f'(z)|^2|f(z)|^{p-2}\,dA(z) &\le \frac{p}{2\pi}\int_{|w|\le M_\infty(r,f)} |w|^{p-2}\,dA(w) \\
&= p\int_0^{M_\infty(r,f)} t^{p-1}\,dt = M_\infty^p(r,f),
\end{aligned}
\tag{3.2}
$$

in particular,

$$
\frac{1}{\pi}\int_{\Delta(0,r)} |f'(z)|^2\,dA(z) \le M_\infty^2(r,f). \tag{3.3}
$$

Inequality (3.3) yields

$$
\|f\|_{S_\alpha^p}^p \le \pi^{\frac{p}{2}}\int_0^1 M_\infty^p(r,f)(1-r^2)^{\alpha+1}\,dr = \pi^{\frac{p}{2}} J_\alpha^p(f).
$$

3.3. Proof of (2) ⇒ (1)

This implication is proved by following the reasoning in [23, pp. 890-891]. Let $2\le p<\infty$ and $-2<\alpha<\infty$ such that $p+\alpha>-1$, and let $f\in\mathcal{H}(\mathbb{D})$. Then, by (2.1),

$$
\begin{aligned}
|f'(0)|^p &\le \left(\|f\|_{H^2}^2 - |f(0)|^2\right)^{\frac{p}{2}} \le \|f\|_{H^2}^p - |f(0)|^p \le \|f\|_{H^p}^p - |f(0)|^p \\
&= \frac{p^2}{2}\int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2 \log\frac{1}{|z|}\,dA(z).
\end{aligned}
$$

An application of this inequality to the function $f(z/2)$ gives

$$
|f'(0)|^p \le 2^{p-1}p^2\int_{\Delta(0,\frac{1}{2})} |f(z)|^{p-2}|f'(z)|^2 \log\frac{1}{2|z|}\,dA(z).
$$

Replace now f by $f\circ\varphi_w$ to obtain

$$\begin{aligned}|f'(w)|^p(1-|w|^2)^p &\le 2^{p-1}p^2\int_{\Delta(0,\frac12)}|(f\circ\varphi_w)(z)|^{p-2}|(f\circ\varphi_w)'(z)|^2\log\frac{1}{2|z|}\,dA(z)\\ &= 2^{p-1}p^2\int_{D(w,\frac12)}|f(v)|^{p-2}|f'(v)|^2\log\frac{1}{2|\varphi_w(v)|}\,dA(v),\end{aligned}$$

where $D(w,\frac12):=\{z:|\varphi_w(z)|<\frac12\}$ is a pseudohyperbolic disc. Integrating this inequality with respect to $(1-|w|^2)^\alpha\,dA(w)$ and applying Fubini's theorem, it follows that

$$\begin{aligned}\|f\|^p_{\mathcal{D}^p_{p+\alpha}} &\le 2^{p-1}p^2\int_{\mathbb{D}}\int_{D(w,\frac12)}|f(v)|^{p-2}|f'(v)|^2\log\frac{1}{2|\varphi_w(v)|}\,dA(v)(1-|w|^2)^\alpha\,dA(w)\\ &= 2^{p-1}p^2\int_{\mathbb{D}}|f(v)|^{p-2}|f'(v)|^2\int_{D(v,\frac12)}\log\frac{1}{2|\varphi_w(v)|}(1-|w|^2)^\alpha\,dA(w)\,dA(v).\end{aligned}$$

But

$$\begin{aligned}\int_{D(v,\frac12)}\log\frac{1}{2|\varphi_w(v)|}(1-|w|^2)^\alpha\,dA(w) &= \int_{\Delta(0,\frac12)}\log\frac{1}{2|z|}\frac{(1-|v|^2)^{2+\alpha}(1-|z|^2)^\alpha}{|1-\bar{v}z|^{4+2\alpha}}\,dA(z)\\ &\lesssim (1-|v|^2)^{2+\alpha},\end{aligned}$$

and therefore (2.3) yields $\|f\|_{\mathcal{D}^p_{p+\alpha}}\lesssim\|f\|_{H^p_\alpha}$.

It is worth noticing that $A^p_\alpha=\mathcal{D}^p_{p+\alpha}$ for $\alpha>-1$, and therefore the proof above shows one of the asymptotic inequalities in (2.2).

4. Proof of Theorem 3

It will be shown that (1) ⇒ (3) ⇒ (2) ⇒ (1).

4.1. Proof of (1) ⇒ (3)

Let $0<p<1$ and $-2<\alpha<\infty$ such that $p+\alpha>-1$. By Lemma I,

$$M_1^p(r,f')\lesssim M_p^p\left(\frac{1+r}{2},f'\right)(1-r)^{p-1},$$

and it follows that

$$\begin{aligned}K^p_\alpha(f) &\lesssim \int_0^1 M_p^p\left(\frac{1+r}{2},f'\right)(1-r^2)^{p+\alpha}\,dr\\ &\lesssim \int_0^1 M_p^p(\rho,f')(1-\rho^2)^{p+\alpha}\,d\rho\simeq\|f\|^p_{\mathcal{D}^p_{p+\alpha}}.\end{aligned}$$

4.2. Proof of (3) $\Rightarrow$ (2)

This implication holds for all $0 < p < \infty$. Let $0 < p < \infty$ and $-2 < \alpha < \infty$, and let $f \in \mathcal{U}$. The image of the circle of radius r centered at the origin under the function $f - f(0)$ is a Jordan curve with zero in its inner domain. The length of this image is $2\pi r M_1(r, f')$, and it follows that

$$M_\infty(r, f) \leq \pi r M_1(r, f') + |f(0)|. \tag{4.1}$$

This yields $J_\alpha^p(f) \lesssim K_\alpha^p(f) + |f(0)|^p$ as desired.

4.3. Proof of (2) $\Rightarrow$ (1)

This implication is proved by following step-by-step the corresponding proof in [7]. Since the proof is rather lengthy and technical, yet almost identical with the original one, the parts which differ from the original reasoning are indicated only. By using the same notation as in [7, pp. 842-847], the differences are as follows. Let $0 < p < 1$ and $-\frac{475}{316} < \alpha < -1$ such that $p > -\frac{316}{159}(\alpha + 1)$, and let $f \in \mathcal{U}$ be zero free such that $f(0) = 1$. First, replace equation (14) by

$$\int_{A_j(r)\cap I} |f'(re^{i\theta})|^p (1-r)^{p+\alpha}\, dr \leq C 2^{-jp} M^p(r) \left(\frac{1-\rho}{1-r}\right)^{\beta p - p + 1} (1-r)^{\alpha+1}.$$

Then (15) becomes

$$\int_{A_j(r)} |f'(re^{i\theta})|^p (1-r)^{p+\alpha}\, dr \leq C\nu(j, r) 2^{-jp} M^p(r) \left(\frac{1-\rho}{1-r}\right)^{\beta p - p + 1} (1-r)^{\alpha+1},$$

where $0 \leq j < k(r)$. In (17), keep the notation $\gamma(p) \equiv \beta p + 1 - p + \eta$, but choose η such that $\gamma(p) + \alpha + 1 < 1$. This can be done since $0 < p < 1$ and $-\frac{475}{316} < \alpha < -1$ such that $p > -\frac{316}{159}(\alpha + 1)$. Now (19) should be replaced by

$$\int_{A_j(r)} |f'(re^{i\theta})|^p (1-r)^{p+\alpha}\, dr \leq C M^p(r) 2^{-jp} \left(\frac{1-\rho}{1-r}\right)^{\gamma(p)} (1-r)^{\alpha+1},$$

where $\frac{1}{2} < r < 1$, $0 \leq j < k(r)$ and $\rho = \rho(j, k)$, and the analogue of (20) is

$$\begin{aligned}\int_\Delta |f'(z)|^p (1-|z|^2)^{p+\alpha}\, dA(z) = &\int_{\Delta(0,\frac{1}{2})} |f'(z)|^p (1-|z|^2)^{p+\alpha}\, dA(z)\\ &+ \int_B |f'(z)|^p (1-|z|^2)^{p+\alpha}\, dA(z)\\ &+ \int_A |f'(z)|^p (1-|z|^2)^{p+\alpha}\, dA(z).\end{aligned}$$

Clearly,

$$\int_{\Delta(0,\frac{1}{2})} |f'(z)|^p(1-|z|^2)^{p+\alpha}\,dA(z)<\infty.$$

Inequality (23) should be replaced by

$$\int_B |f'(z)|^p(1-|z|^2)^{p+\alpha}\,dA(z)\le C\int_{\frac{1}{2}}^1 (1-r)^{-\beta p+p+\alpha}\,dr,$$

where the last integral converges since $-\beta p+p+\alpha>-1$ by the assumption $p>-\frac{316}{159}(\alpha+1)$ (recall that $\beta=\frac{1}{2}-\frac{1}{316}$). It remains to consider the last term. Following the proof further, replace (26) by

$$\int_A |f'(z)|^p(1-|z|^2)^{p+\alpha}\,dA(z)\le C\sum_{j=0}^{\infty}\int_{r_j}^1 M^p(\rho)\left(\frac{1-\rho}{1-r}\right)^{\gamma(p)}(1-r)^{\alpha+1}\,dr,$$

and then (30) becomes

$$\int_{r_j}^1 M^p(\rho)\left(\frac{1-\rho}{1-r}\right)^{\gamma(p)}(1-r)^{\alpha+1}\,dr\le C\sum_{m=m_0}^{\infty}M^p(1-b^m)b^{m\gamma(p)}\int_{E_{jm}}(1-r)^{-\gamma(p)+\alpha+1}\,dr.$$

Moreover, (31) becomes

$$\begin{aligned}\int_A |f'(z)|^p(1-|z|^2)^{p+\alpha}\,dA(z)&\le C\sum_{m=m_0}^{\infty}M^p(1-b^m)b^{m\gamma(p)}\int_{1-b^m}^1(1-r)^{-\gamma(p)+\alpha+1}\,dr\\&=C\sum_{m=m_0}^{\infty}M^p(1-b^m)b^{-\alpha m}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{1-b^m}^{1-b^{m+1}} M^p(r)(1-r)^{\alpha+1}\,dr&\ge M^p(1-b^m)\frac{b^{m(\alpha+2)}}{\alpha+2}\left(1-b^{\alpha+2}\right)\\&\ge M^p(1-b^m)\frac{b^{-\alpha m}}{\alpha+2}\left(1-b^{\alpha+2}\right),\end{aligned}$$

and it follows that

$$\int_A |f'(z)|^p(1-|z|^2)^{p+\alpha}\,dA(z)\le C\int_0^1 M^p(r)(1-r)^{\alpha+1}\,dr.$$

A careful inspection on the original proof with the modifications presented here shows that

$$\|f\|^p_{\mathcal{D}^p_{p+\alpha}}+|f(0)|^p\lesssim J^p_\alpha(f)\tag{4.2}$$

for $f\in\mathcal{U}$ under the assumptions $0<p<1$ and $-\frac{475}{316}<\alpha<-1$ such that $p>-\frac{316}{159}(\alpha+1)$. This finishes the proof.

5. Proof of Theorem 4

The cases $0 < p < 2$ and $2 \le p < \infty$ will be considered separately.

The case $0 < p < 2$

It will be shown that (1) $\Rightarrow$ (4) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). The only implication which uses the univalence of f is (4) $\Rightarrow$ (2).

5.1. Proof of (1) $\Rightarrow$ (4)

Let $f \in \mathcal{B}$. An application of the inequality

$$M_\infty(r,g) \le \|g\|_{\mathcal{B}} \frac{1}{2} \log \frac{1+r}{1-r} + |g(0)|, \quad g \in \mathcal{B},$$

to the function $g = f \circ \varphi_a - f(a) \in \mathcal{B}$ gives

$$M_\infty(r, f\circ\varphi_a - f(a)) \le \|f\circ\varphi_a - f(a)\|_{\mathcal{B}} \frac{1}{2}\log\frac{1+r}{1-r} = \|f\|_{\mathcal{B}}\frac{1}{2}\log\frac{1+r}{1-r}. \quad (5.1)$$

This yields

$$\sup_{a\in\mathbb{D}} J^p_\alpha(f\circ\varphi_a - f(a)) \lesssim \|f\|^p_{\mathcal{B}} \int_0^1 \left(\log\frac{1+r}{1-r}\right)^p (1-r^2)^{\alpha+1}\,dr < \infty.$$

5.2. Proof of (4) $\Rightarrow$ (2)

Inequality (3.2) yields

$$\|f\|^p_{H^p_\alpha} \le \frac{2\pi}{p}\int_0^1 M^p_\infty(r,f)(1-r^2)^{\alpha+1}\,dr = \frac{2\pi}{p} J^p_\alpha(f), \quad (5.2)$$

and the assertion follows by applying this inequality to the function $f\circ\varphi_a - f(a)$.

5.3. Proof of (2) $\Rightarrow$ (3)

The proof of [24, Theorem 2] shows that $\|f\|_{S^p_\alpha} \lesssim \|f\|_{H^p_\alpha}$ and the assertion follows by applying this asymptotic inequality to the function $f\circ\varphi_a - f(a)$.

5.4. Proof of (3) $\Rightarrow$ (1)

Let $0 < p < \infty$ and $-2 < \alpha < \infty$, and let $f \in \mathcal{H}(\mathbb{D})$. Then, by the change of variable $w = \varphi_a(z)$ and the sub-mean-value property of $|f|^2$,

$$\begin{aligned}
\|f \circ \varphi_a - f(a)\|_{S_\alpha^p}^p &= \int_0^1 r(1-r^2)^{\alpha+1} \left(\int_{\Delta(0,r)} |f'(\varphi_a(z))|^2 |\varphi_a'(z)|^2 \, dA(z) \right)^{\frac{p}{2}} dr \\
&\geq \int_{\frac{1}{2}}^1 r(1-r^2)^{\alpha+1} \left(\int_{D(a,\frac{1}{2})} |f'(w)|^2 \, dA(w) \right)^{\frac{p}{2}} dr \\
&\gtrsim \int_{\frac{1}{2}}^1 r(1-r^2)^{\alpha+1} \, dr \left(|f'(a)|(1-|a|^2) \right)^p \\
&\simeq \left(|f'(a)|(1-|a|^2) \right)^p ,
\end{aligned}$$

and it follows that $f \in \mathcal{B}$ if (3) is satisfied.

The case $2 \leq p < \infty$

By the proof of Theorem 1 the conditions (2), (3) and (4) are equivalent. Moreover, (2) $\Rightarrow$ (1) follows by (2.12), the asymptotic inequality $\|f\|_{\mathcal{D}_{p+\alpha}^p} \lesssim \|f\|_{H_\alpha^p}$ established in Section 3.3 and the fact $\|f\|_{\mathcal{B}} \lesssim \|f\|_{B_{2+\alpha}^p}$ which holds for all $0 < p < \infty$ and $-2 < \alpha < \infty$. The proof is completed by showing that (1) implies (4).

5.5. Proof of (1) $\Rightarrow$ (4)

Let $0 < p < \infty$ and $-2 < \alpha < \infty$, and let $f \in \mathcal{B} \cap \mathcal{U}$. By (3.3) and (5.1),

$$\sup_{a \in \mathbb{D}} \int_{\Delta(0,r)} |(f \circ \varphi_a)'(z)|^2 \, dA(z) \lesssim \|f\|_{\mathcal{B}}^2 \left(\log \frac{1+r}{1-r} \right)^2 , \tag{5.3}$$

and it follows that

$$\sup_{a \in \mathbb{D}} \|f \circ \varphi_a - f(a)\|_{S_\alpha^p}^p \lesssim \|f\|_{\mathcal{B}}^p \int_0^1 \left(\log \frac{1+r}{1-r} \right)^p (1-r^2)^{\alpha+1} \, dr < \infty.$$

6. Proof of Theorem 7

This is a consequence of Theorem 4 and the proofs of Theorems 1 and 3. Namely, by (2.12) and (3.1), $\sup_{a \in \mathbb{D}} J_\alpha^p(f \circ \varphi_a - f(a)) \lesssim \|f\|_{B_{2+\alpha}^p}$, and by (4.2), $\|f\|_{B_{2+\alpha}^p} \lesssim \sup_{a \in \mathbb{D}} J_\alpha^p(f \circ \varphi_a - f(a))$. The assertion now follows by Theorem 4.

7. Proof of Theorem 9

A careful observation on the proof of Theorem 4 shows that it suffices to show the implications (1) ⇒ (3) and (1) ⇒ (4).

7.1. Proof of (1) ⇒ (3)

Let $0 < p < \infty$ and $-2 < \alpha < \infty$, and let $f \in \mathcal{B}_0 \cap \mathcal{U}$. Inequality (5.3) implies

$$\begin{aligned}\|f\circ\varphi_a - f(a)\|_{S^p_\alpha}^p &\lesssim \|f\|_{\mathcal{B}}^p \int_R^1 r(1-r^2)^{\alpha+1}\left(\log\frac{1+r}{1-r}\right)^p dr\\ &\quad + \int_0^R r(1-r^2)^{\alpha+1}\left(\int_{\Delta(0,r)} |(f\circ\varphi_a)'(z)|^2\,dA(z)\right)^{\frac{p}{2}} dr\\ &=: I_1(f) + I_2(f)\end{aligned}$$

for all $0 < R < 1$. Let $\varepsilon > 0$. Fix R such that $I_1(f) < \frac{\varepsilon^p}{2}$. Then

$$I_2(f) \lesssim \left(\int_{\Delta(0,R)} |f'(\varphi_a(z))|^2(1-|\varphi_a(z)|^2)^2\,dA(z)\right)^{\frac{p}{2}},$$

and since $f \in \mathcal{B}_0$, it follows that for any given $\varepsilon_1 > 0$ there is a $R_{\varepsilon_1} \in (0,1)$ such that $|f'(\varphi_a(z))|(1-|\varphi_a(z)|^2) < \varepsilon_1$ for all $|z| \le R$ when $|a| > R_{\varepsilon_1}$. Therefore there exists an $R_1 \in (0,1)$ such that $I_2(f) < \frac{\varepsilon^p}{2}$ for $|a| > R_1$, and it follows that $\|f\circ\varphi_a - f(a)\|_{S^p_\alpha}^p < \varepsilon^p$ for $|a| > R_1$. Thus (3) is satisfied if $f \in \mathcal{B}_0 \cap \mathcal{U}$.

7.2. Proof of (1) ⇒ (4)

Let $f \in \mathcal{B}_0$ and $\varepsilon > 0$. By the inequality (5.1) there is an $R \in (0,1)$ such that

$$\int_R^1 M_\infty^p(r, f\circ\varphi_a - f(a))(1-r^2)^{\alpha+1}\,dr < \frac{\varepsilon}{2} \tag{7.1}$$

for all $a \in \mathbb{D}$. Moreover, by (4.1),

$$\begin{aligned}\int_0^R M_\infty^p(r, f\circ\varphi_a - f(a))(1-r^2)^{\alpha+1}\,dr &\lesssim \int_0^R M_1^p(r,(f\circ\varphi_a)')(1-r^2)^{\alpha+1}\,dr\\ &\lesssim \int_0^R\left(\int_0^{2\pi} |f'(\varphi_a(re^{i\theta}))|(1-|\varphi_a(re^{i\theta})|^2)\right)^p dr,\end{aligned}$$

and it follows that there is an $R_\varepsilon \in (0,1)$ such that

$$\int_0^R M_\infty^p(r, f\circ\varphi_a - f(a))(1-r^2)^{\alpha+1}\,dr < \frac{\varepsilon}{2} \tag{7.2}$$

for all $|a| > R_\varepsilon$. The inequalities (7.1) and (7.2) imply that (4) is satisfied if $f \in \mathcal{B}_0$.

8. Proof of Theorem 10

This is a consequence of Theorem 9 and the proofs of Theorems 1 and 3. Namely, by (2.12) and (3.1), $\lim_{|a|\to 1^-} J_\alpha^p(f\circ\varphi_a - f(a)) \lesssim \lim_{|a|\to 1^-} \|f\circ\varphi_a\|_{\mathcal{D}^p_{p+\alpha}}$, and by (4.2), $\lim_{|a|\to 1^-} \|f\circ\varphi_a\|_{\mathcal{D}^p_{p+\alpha}} \lesssim \lim_{|a|\to 1^-} J_\alpha^p(f\circ\varphi_a - f(a))$. Since the change of variable $w=\varphi_a(z)$ shows that $\lim_{|a|\to 1^-} \|f\circ\varphi_a\|_{\mathcal{D}^p_{p+\alpha}} = 0$ if and only if $f\in B^p_{2+\alpha,0}$, the assertion follows by Theorem 9.

9. Proof of Theorem 11

The implication (1) ⇒ (2) is equivalent to [6, Theorem 1]. Moreover, (2) implies (1) by Section 4.2, and it is known that (3) implies (1), see [44]. It remains to show that (2) implies (3).

9.1. Proof of (2) ⇒ (3)

This can be done by using Carleson measures. For $s>0$, a positive measure μ on $\mathbb{D}$ is said to be a bounded s-Carleson measure if $\mu(S(I)) \lesssim |I|^s$, where $|I|$ denotes the arc length of a subarc I of the boundary of $\mathbb{D}$ and $S(I) = \left\{z\in\mathbb{D} : \frac{z}{|z|}\in I, 1-|I|\le |z|\right\}$ is the Carleson box based on I. These measures (for $s=1$) were introduced by Carleson [8, 9], see [26] for a list of relevant references. To prove (2) implies (3), it suffices to consider the case $p=1$ since $\|f\|_{B^p_{\alpha,s}} \lesssim \|f\|_{B^1_{\alpha,s}}$ for all $1<p<\infty$. By [3, Lemma 2.1] and the assumption (2),

$$\begin{aligned}\|f\|^p_{B^p_{\alpha,s}} &\simeq \sup_I \frac{1}{|I|^s}\int_{S(I)} |f'(z)|(1-|z|^2)^{\alpha-2+s}\,dA(z)\\ &\lesssim \sup_I \frac{1}{|I|^s}\int_{1-|I|}^1 M_1(r,f')(1-r)^{\alpha-2+s}\,dr\\ &\lesssim \sup_I \frac{1}{|I|^s}\int_{1-|I|}^1 (1-r)^{s-1}\,dr \le \frac{1}{s},\end{aligned}$$

and it follows that $f\in B^p_{\alpha,s}$ if (3) is satisfied.

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